1. Let p be a prime number. Define $\mathbb{Z}_p$, the set of residue classes modulo p. Prove that $\mathbb{Z}_p^*$, the set of nonzero members of $\mathbb{Z}_p$, is a group under multiplication.

Now suppose q is a prime, and p is an odd prime which divides $7^q - 1$. Prove that either $p = 3$ or q divides $p - 1$.

Find all positive integers less than 50 which divide $7^{11} - 1$.

2. Let G be a group, with identity element e.

Prove that e is the unique identity element of G, and that each element of G has a unique inverse.

Suppose now that G is a finite abelian group. Let z denote the product of all the elements in G. Prove that $z^2 = e$.

Deduce that if p is a prime number then p divides $((p-1)!^2 - 1)$. (You may assume that $\{1, 2, ..., p-1\}$ is a group under multiplication modulo p.)

Let n be an integer, with $n > 4$, such that n is not a prime number.

Prove that n divides $(n-1)!$. Deduce that n does not divide $((n-1)!^2 - 1)$.

3. Let A, B be the following 2×2 matrices over $\mathbb{C}$:

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(i) Show that $A^4 = I$, $A^2 = B^2$ and $BA = A^{-1}B$.

(ii) Let

$$Q = \{A^r B^s \mid \text{ all } r, s \in \mathbb{Z}\}.$$

Prove that $|Q| = 8$, and Q is a group.

Now let G be a group of order 8, containing elements x, y, both of order 4, with $y \notin \langle x \rangle$. Assume G is non-abelian.

(a) Prove that $yx = x^{-1}y$.

(b) Prove that $y^2 = x^2$.

(c) Deduce that $G \cong Q$.

4. Let H be a subgroup of a finite group G. What is a right coset of H in G? What is meant by the statement that H is normal in G?

Assume now that H is normal in G. Explain how to define a binary operation on the set G/H of right cosets H in G in such a way that G/H becomes a group. Prove that G/H is, indeed, a group with this binary operation. (If you assume the result that $HH = H$ then you should prove this result.)

Show that if G/H has order 2 the every element of G either belongs to H or has even order.

5. Let G and H be finite groups and let ϑ be a homomorphism from G to H. Define the <u>kernel</u> of ϑ and the <u>image</u> of ϑ. Prove that the kernel of ϑ is a subgroup of G and state (without proof) a result which relates the order of G, the order of the kernel of ϑ and the order of the image of ϑ.

You may assume (without proof) that there is a homomorphism ϕ from S_4 onto S_3 which satisfies

$$\phi((12)) = (23), \quad \phi((23)) = (12), \quad \phi((34)) = (23).$$

(i) Prove that $(12)(34)$ belongs to the kernel of ϕ.

(ii) Write $(12)(23)(12)$ and $(23)(34)(23)$ as product of disjoint cycles.

(iii) Find $\phi((13)(24))$.

(iv) List the elements which are in the kernel of ϕ. Justify your answer.