Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2007

This paper is also taken for the relevant examination for the Associateship.

M2P1

Analysis II

Date: examdate Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (i) Let $f : \mathbb{R} \to \mathbb{R}$. Write down the definition of "f is continuous at $b \in \mathbb{R}$" in terms of ϵ, δ.

(ii) Let $g : \mathbb{R}^2 \to \mathbb{R}$. Write down the definition of "g is continuous at $a \in \mathbb{R}^2$" in terms of ϵ, δ.

(iii) Let us define functions $h_1, h_2 : \mathbb{R}^2 \to \mathbb{R}$ by

$$h_1(x_1, x_2) = x_1 \quad \text{and} \quad h_2(x_1, x_2) = x_2.$$

Prove (using the definition in (ii), or otherwise) that h_1 and h_2 are continuous at any $a \in \mathbb{R}^2$.

(iv) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at all points of $\mathbb{R}$. Define $F : \mathbb{R}^2 \to \mathbb{R}$ by

$$F(x_1, x_2) = f(x_1 + x_2).$$

Prove that F is continuous at all points of $\mathbb{R}^2$.

[In (iv) state explicitly any results on continuous functions that you use]

2. Let $f, g : \mathbb{R} \to \mathbb{R}$. State the Chain Rule Theorem for the composition $f \circ g$.

(i) Suppose that g is twice differentiable at a and that f is twice differentiable at $g(a)$. Prove that the composition $f \circ g$ is twice differentiable at a. Assuming that $g'(a) = 0$, show that

$$(f \circ g)''(a) = f'(g(a))g''(a).$$

(ii) Let f be strictly increasing and twice differentiable on $\mathbb{R}$. Let $a \in \mathbb{R}$ and let $b = f(a)$. Assuming that $f'(a) \neq 0$, prove that the inverse function f^{-1} is twice differentiable at b and show that

$$(f^{-1})''(b) = -\frac{f''(a)}{(f'(a))^3}.$$

[In Question 2, you may use the chain rule and the inverse function theorem without justification]

3. (i) State Rolle's theorem and prove it.

(ii) Let f be continuous on $[a, b]$, differentiable on (a, b). Assume that $f(a) = f(b)$ and that f is not a constant function. Prove that there exist some points $x_1, x_2 \in [a, b]$ such that $f'(x_1) > 0$ and $f'(x_2) < 0$.

(iii) Give an example of a function f which is continuous on $[0, 1]$, differentiable on $(0, 1)$, and which does not have a right derivative at 0.

[In (iii), it is enough to give the precise definition of such a function without proofs]

4. State L'Hôpital's rule (right-sided version).

(i) Let $a \in \mathbb{R}$. Let f be twice differentiable around a and assume that $f(a) = 0$. Let us define function F by
$$F(x) = f(x) - f'(a)(x - a).$$
Prove that
$$\frac{F(x)}{x - a} \to 0 \quad \text{as} \quad x \to a.$$
[If you use L'Hôpital's rule in this part, you must verify all of its assumptions]

(ii) Let $a \in \mathbb{R}$. Let f and g be twice differentiable around a and assume that $f(a) = g(a) = 0$. Assume in addition that $g'(a) \neq 0$. Prove (using (i), or otherwise) that
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

5. Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

(i) Give definitions of upper and lower Riemann sums and of upper and lower Riemann integrals of f on $[a, b]$.

(ii) Prove that if f is Riemann integrable on $[a, b]$ and if $f(x) \geq 0$ for all $x \in [a, b]$, then $\int_a^b f \geq 0$.

(iii) Prove that if f and g are Riemann integrable on $[a, b]$ and if $f(x) \geq g(x)$ for all $x \in [a, b]$, then $\int_a^b f \geq \int_a^b g$.

(iv) Let f and g be Riemann integrable on $[a, b]$. Prove that $\int_a^b |f| + \int_a^b |g| \geq \int_a^b (f + g)$.

[In Question 5 you may use the linearity of the Riemann integral without justification]