Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M2P1

Analysis II

Date: Wednesday, 10 May 2006 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Let $f, g : (-1, 1) \to \mathbb{R}$.

(i) Write down the definition of "$f(x) \to 5$ as $x \to 0$" in terms of ϵ, δ.

(ii) Let $g(0) = \alpha$ for some $\alpha \in \mathbb{R}$. Write down the definition of "$g(x)$ is continuous at 0" in terms of ϵ, δ.

Let now $f(x) \to 5$ as $x \to 0$, $g(0) = \alpha$, and let $g(x)$ be continuous at 0.

(iii) Suppose that f is not continuous at 0 and that $\alpha \neq 0$. Prove that the product $f(x)g(x)$ is not continuous at 0.

(iv) Suppose that f is not continuous at 0 and that $\alpha = 0$. Is the product $f(x)g(x)$ continuous at 0? Give a proof or a counterexample.

2. State the Intermediate Value Theorem.

Let now $f : [-1, 1] \to \mathbb{R}$ satisfy $f(x) = -f(-x)$ for all $x \in [-1, 1]$. Suppose also that f is continuous on $[-1, 0]$.

(i) Prove that f is continuous on $[-1, 1]$.

(ii) Suppose that there is $a \in [-1, 1]$ such that $f(a) = 2006$. Prove that there exists $b \in [-1, 1]$ such that $f(b) = 2005$.

(iii) Suppose in addition that f is strictly increasing on $[-1, 0]$. Prove that $f : [-1, 1] \to \mathbb{R}$ has a strictly increasing inverse function $g : [-f(1), f(1)] \to \mathbb{R}$, continuous on $[-f(1), f(1)]$. (Here you may use the Inverse Function Theorem without justification.)

(iv) Suppose in addition to (iii) that f is differentiable on $(-1, 1)$. Does it follows that its inverse $g : [-f(1), f(1)] \to \mathbb{R}$ is differentiable on $(-f(1), f(1))$? Give a proof or a counterexample.

3. (i) Write down the definition of "f is left differentiable at a" and define the left derivative $f'_{-}(a)$.

(ii) Show that if f is left differentiable at a, then it is left continuous at a.

Let f be left differentiable at a. Define $F(x) = f(x)$ for $x \leq a$ and $F(x) = \alpha x + \beta$ for $x > a$, for some $\alpha, \beta \in \mathbb{R}$.

(iii) Determine values α, β for which F is continuous and differentiable at a.

(iv) What do we have to assume of f for F to be twice differentiable at a? Give reasons for your answer.

4. State the Mean Value Theorem and prove it. In the proof, you may use Rolle's theorem without justification.

Let $-\infty < a < b < \infty$ and let f be differentiable on (a, b).

(i) Prove that if $f(x)$ is not bounded on (a, b), then its derivative $f'(x)$ is not bounded on (a, b).

(ii) Show that the converse to (i) does not hold, i.e. $f(x)$ may be bounded on (a, b), while $f'(x)$ is not bounded on (a, b).

5. Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

(i) Formulate the ϵ-criterion for Riemann integrability of f.

(ii) Prove that if f is Riemann integrable, then for every $\epsilon > 0$ there is a partition Δ of $[a, b]$ such that $S(f, \Delta) - s(f, \Delta) < \epsilon$.

(iii) Assume that f is continuous on $[a, b]$. Let $\alpha \in \mathbb{R}$. Show that the function

$$F(x) = \int_a^x \left(\int_a^y f(z)dz \right) dy + \alpha$$

is continuous on $[a, b]$. Show also that F and F' are differentiable on (a, b). (Here you may use the preparation theorem for the Fundamental Theorem of Calculus without proof.)

(iv) Let F from (iii) satisfy $F(x) = 2006$ for all $x \in [a, b]$. Show that $f(x) = 0$ for all $x \in [a, b]$ and that $\alpha = 2006$.