UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2005

This paper is also taken for the relevant examination for the Associateship.

M2P1 ANALYSIS II

Date: Monday 10th May 2005 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Write down the definitions of "$f(x)$ is differentiable at a" and of $f'(a)$.

(i) Let $f(x) = x(x-1)\cdots(x-2005)$. Find $f'(0)$.

(ii) Let $f(x) = x^2$ for $x \in \mathbf{Q}$ and $f(x) = 0$ otherwise. Show that f is differentiable at a if and only if $a = 0$.

(iii) Let $f(x) = |x-a|\phi(x)$, where $\phi(x)$ is continuous and $\phi(a) \neq 0$. Prove that f is not differentiable at a. Find the right and left derivatives $f'_+(a)$ and $f'_-(a)$.

2. State the Inverse Function Theorem.

(i) Prove that the function $f(x) = x^{2/3}$ is continuous at all points $a \in \mathbf{R}$.

(ii) Prove that there exists a uniquely determined function $y = y(x)$ satisfying the equation

$$y^3 + 3y = x.$$

Prove that this function $y(x)$ is strictly increasing.

(iii) Find the derivative $y'(x)$ of the function $y(x)$ from (ii).

3. Let $f, g : \mathbf{R} \to \mathbf{R}$ be two functions differentiable at all points. Assume that $f(x) > 0$ for all $x \in \mathbf{R}$. Let us define

$$h(x) = [f(x)]^{g(x)} = E(g(x)\log f(x))$$

($f(x)$ raised to the power $g(x)$).

(i) Prove that h is differentiable at all points and calculate its derivative in terms of f, g and their derivatives.

(ii) Assume further that $f(x) = 1 + kx + u(x)$ where $\frac{u(x)}{x} \to 0$ as $x \to 0$. Calculate $f(0)$ and $f'(0)$.

(iii) Let $f(x)$ be as in (ii). Prove that $\lim_{x\to 0}[f(x)]^{1/x} = e^k$.

4. State Rolle's theorem and prove it.

Now let $f : [0, 2] \to \mathbf{R}$ be right continuous at $a = 0$ and left continuous at $a = 2$. Also, assume that f has two derivatives f', f'' on $(0, 2)$.

(i) Prove that f is continuous on the closed interval $[0, 2]$.

(ii) Suppose further that $f(0) = f(1) = f(2) = 0$. Prove that there is a point $c \in (0, 2)$ such that $f''(c) = 0$.

(iii) Given in addition that $|f'(x)| \leq 1$ for all $x \in (0, 2)$, prove that $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in [0, 2]$.

5. Let $f, g : [a, b] \to \mathbf{R}$ be two bounded functions.

(i) Give definitions of upper and lower Riemann sums $S(f, \Delta), s(f, \Delta)$, and upper and lower Riemann integrals $J(f), j(f)$ of f on $[a, b]$.

(ii) Prove (directly from definitions in (i)) that $J(f + g) \leq S(f, \Delta) + S(g, \Delta)$ for all partitions Δ of $[a, b]$.

(iii) Let $f, g : [a, b] \to \mathbf{R}$ be Riemann integrable on $[a, b]$. Show that $f + g$ is also Riemann integrable on $[a, b]$ and that $\int_a^b (f + g) = \int_a^b f + \int_a^b g$. Here in part (iii) you may use without proof other properties of upper and lower Riemann integral from the course.