Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2004

This paper is also taken for the relevant examination for the Associateship.

M2P1 ANALYSIS II

Date: Monday 10th May 2004 Time: 14 pm – 16 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Write down the definition of "$f(x) \to l$ as $x \to a$" in terms of ϵ, δ.

For some $\alpha, \beta \in \mathbf{R}$ define $f(x) = \alpha x$ for $x \in \mathbf{Q}$ and $f(x) = \beta x$ for $x \notin \mathbf{Q}$.

(i) If $\alpha \neq \beta$, find all points $a \in \mathbf{R}$ such that f is continuous at a.

(ii) Find all values of α, β such that f^2 is continuous at all points.

(iii) Let now $\alpha, \beta \in \mathbf{Q}$, $\alpha \neq 0$, $\beta \neq 0$. Find all α, β such that $f \circ f$ is continuous at all points.

2. State the Extreme Value Theorem.

Let $f, g : [0,1] \to \mathbf{R}$ be continuous on $[0,1]$. Assume also that $f(x) \neq 0$ and $g(x) \neq 0$ for all $x \in [0,1]$.

(i) Prove that there is an $\epsilon > 0$ such that $|f(x)| \geq \epsilon$ for all $x \in [0,1]$.

(ii) Prove that there is $\alpha > 0$ such that $f(x) + \alpha g(x) \neq 0$ for all $x \in [0,1]$.

(iii) Suppose in addition that f and g are strictly increasing. Show that for any $\alpha > 0$ there is a function h such that $f(h(y)) + \alpha g(h(y)) = y$, for y in some interval in $\mathbf{R}$. Show also that we can find such h to be defined and continuous at $y = f(1/2) + \alpha g(1/2)$.

3. State the Mean Value Theorem and prove it.

Let $f : [0,1] \to \mathbf{R}$ be right continuous at $a = 0$ and left continuous at $a = 1$. Assume that f' exists on $(0,1)$.

(i) Prove that f is continuous on $[0,1]$.

(ii) Suppose in addition that $f(0) = 0$. Show that for every integer $n \in \mathbf{N}$ there is a point $c_n \in (0,1)$ such that $f'(c_n) = nf(1/n)$.

(iii) Let $g : [0,1] \to \mathbf{R}$ be continuous on $[0,1]$ and twice differentiable on $(0,1)$. Suppose that $g''(x) > 0$ for all $x \in (0,1)$. Prove that g can not have a local maximum in $(0,1)$.

4. State the two-sided version of the Taylor's theorem with Lagrange's form of the remainder.

Let functions f, g be three times differentiable on some open interval I containing $a = 0$. Let $f(0) = g(0) = 0$.

(i) Deduce L'Hôpital's rule for $\lim_{x\to 0} \frac{f(x)}{g(x)}$ from the Taylor's theorem.

(ii) Prove that if $f'(0) \neq 0$, then there is $\epsilon > 0$ such that for all x with $0 < |x| < \epsilon$ we have $|\frac{f(x)}{x}| \leq 2|f'(0)|$.

(iii) Suppose that $|g'(x)| \leq |x|$ for all $x \in I$. Prove that $|g(x)| \leq x^2$ for all $x \in I$.

5. State the Fundamental Theorem of Calculus.

(i) Prove that any decreasing function g on $[a, b]$ is Riemann integrable over $[a, b]$.

(ii) Let f be continuous on $[a, b]$. Show that for every $c \in (a, b)$ we have

$$\lim_{\epsilon \to 0} \frac{\int_0^\epsilon f(c+t)dt}{\epsilon} = f(c).$$

(iii) Let $\alpha \in \mathbf{R}$, let $h : [0, 1] \to \mathbf{R}$ be continuous on $[0, 1]$, and suppose that $\int_a^b h = 2004\alpha$ for all $a, b \in [0, 1]$. Prove that $\alpha = 0$ and $h(x) = 0$ for all $x \in [0, 1]$.