1. Consider the linear programming problem

To maximise $x_1 + x_2 + 2x_3$,
subject to $x_1 + x_2 + x_3 \leq 4$, $2x_1 - x_2 \geq 2$, with $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$.

(i) Express the above as a linear programming problem in standard form using slack variables and extra variables to form a feasible basic set of variables giving a non-degenerate starting vertex for applying the simplex algorithm.

(ii) Use the simplex algorithm as given in the course to obtain an optimal vertex, the optimal values of x_1, x_2 and x_3 and the maximum of the objective function. *Show your working.*

2. (i) (a) Let $x \in \Sigma(b)$, the solution set of $Ax = b$, where A is $m \times n$ and b is $m \times 1$.
What is a *feasible set* S?
Show that a feasible set is convex.

(b) A factory makes 3 products A_1, A_2 and A_3.
The profit from a box of A_1, A_2 and A_3 is £2, £3 and £6 respectively.
The resources used are B_1, B_2, $\cdots$, B_5.
The amounts of each resource needed for a box of each product are given in the table, as well as the amounts of each available for use.

	B_1	B_2	B_3	B_4	B_5
A_1	2	4	2	1	1
A_2	1	1	7	1	3
A_3	3	6	2	8	6
Available	80	120	160	160	160

The manufacturer must choose x_1, x_2 and x_3, the number of boxes of each product to produce.
Write this as a linear programming problem.
Verify that $x_1 = 0$, $x_2 = 18$ and $x_3 = 17$ is a feasible solution.
Put the linear programmming problem into standard form.

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(ii) Let $C_n = (V, E)$ be a cycle of length n, where $n \geq 3$:

$$V = \{i \ : \ 0 \leq i \leq n-1\}, \quad E = \{\{i,j\} \ : \ |i-j| \equiv 1 \bmod n\}.$$

Let A be the adjacency matrix of C_n, let ε be an n-th root of unity, and let $v = (1, \varepsilon, \varepsilon^2, ..., \varepsilon^{n-1})^T$.

Prove that v is an eigenvector of A and calculate the corresponding eigenvalue.

How many distinct eigenvalues of A appear when ε runs through the set of all complex n-th roots of unity?

Let $\bar{C}_n = (\bar{V}, \bar{E})$ be the complementary graph of C_n, so that

$$\bar{V} = V, \quad \bar{E} = \{\{i,j\} \ : \ i \neq j, \{i,j\} \notin E\}.$$

Give with justification the number of isomorphisms of C_n onto $\bar{C}_n$.

3. Let $\Gamma = (V, E)$ be a simple graph.

Define

(i) the valency δ_v of a vertex $v \in V$ in Γ;

(ii) a spanning tree for a graph Γ.

Prove the following assertions:

(iii) the equality $\sum_{v \in V} \delta_v = 2 \cdot |E|$ holds;

(iv) if $\Delta = (U, F)$ is a tree with vertex-set U and edge-set F then $|U| = |F| + 1$;

(v) every tree has at least two ends (an *end* of a tree is a vertex of valency 1);

(vi) Γ contains a Hamiltonian path if and only if Γ possesses a spanning tree with exactly two ends.

4. a. Let n and k be integers with $0 < k < n$. Let $C_n^{(k)} = (V, E)$ be the graph with

$$V = \{i \ : \ 0 \leq i \leq n-1\}, \quad E = \{\{i,j\} \ : \ |i-j| \equiv k \bmod \mathrm{n}\}.$$

State for which values of k and n the graph $C_n^{(k)}$

(i) contains a Hamiltonian cycle;

(ii) contains an Eulerian cycle;

(iii) possesses a bipartition.

Give justified answers to the following:

(iv) What is the chromatic number of $C_n^{(k)}$?

(v) What is the number of connected components of $C_n^{(k)}$?

b. Prove that a connected graph containing no odd cycles is bipartite.

5. Let n be a positive integer, $n \geq 3$, let Ω_n be an n-element set, say $\Omega_n = \{1, 2, ..., n\}$. Let $\Gamma_n = (V, E)$ be a directed graph, where V is the set of all subsets of Ω_n (including the empty set and the whole set Ω_n) and E is the set of ordered pairs (A, B) of subsets of Ω_n such that $B \subset A$ and $|A| = |B| + 1$.

If we turn Γ_n into a network W_n what would be the source and the sink? Why?

Let W_n be the above network, and c_0 being a positive real number. Define the capacity function

$$c : E \to \mathbb{R}$$

inductively by the following rule:

$$c(\Omega_n, A) = c_0 \text{ if } |A| = n - 1;$$

if $(A, B), (B, C) \in E$ then

$$c(B, C) = c(A, B) \cdot \frac{n - |B|}{|B|}.$$

Construct a maximal flow in W_4 and calculate its value. Confirm that your flow is maximal by finding the capacity of an appropriate cut.

Construct with justification a maximal flow in W_n for an arbitrary n.