Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M2OD

Graphs, Algorithms and Optimisation

Date: Monday, 15th May 2006 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Consider the linear programming problem

To maximise $5x_1 - 2x_2 + 3x_3$,
subject to $x_1 + x_2 + x_3 = 8, \quad 3x_1 + 2x_2 - x_3 \leq 12,$
with $x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$

(i) Express the above as a linear programming problem in standard form using slack variables and extra variables as needed to form a feasible basic set of variables giving a non-degenerate starting vertex for applying the simplex algorithm.

(ii) Use the simplex algorithm as given in the course to obtain an optimal vertex and the optimal value of the objective function. *Show your working.*

2. (i) Consider the constant-sum game between two players A and B who each have three pure strategies with the matrix of payoffs

$$G = [g_{ij}] = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{ccc} b_1 & b_2 & b_3 \end{array} \\ \begin{pmatrix} 0 & 1 & 2 \\ 5 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix} \end{array}$$

The payment by B to A for simultaneous choices (a_i, b_j) is g_{ij}. Player A seeks to maximise the payment by B, and B seeks to minimise the payment to A.

Consider the randomised strategies whereby B chooses from $\{b_1, b_2, b_3\}$ with probabilities $\{x_1, x_2, x_3\}$ respectively, where $x_1 + x_2 + x_3 = 1$. Similarly A chooses from $\{a_1, a_2, a_3\}$ with probabilities $\{y_1, y_2, y_3\}$ respectively where $y_1 + y_2 + y_3 = 1$.

(a) Express this as a pair of linear programming problems.

(b) Write these two linear programming problems in standard form.

(c) Explain briefly, without applying the simplex algorithm, why the randomised strategies that give the optimal solution are $\{\frac{1}{3}, \frac{1}{6}, \frac{1}{2}\}$ for A and $\{\frac{1}{6}, \frac{1}{3}, \frac{1}{2}\}$ for B. Give the optimal value.

(ii) Let v be a vertex of a graph G. Define the *connected component* $G(v)$ of v.

For $n > 1$, give an example of each of the following.

(a) A graph with n vertices and one connected component;

(b) A graph with n vertices and n connected components.

We define a graph G to have vertices vectors $v = (v_1, v_2, v_3)$ where $v_i \in \{0, 1, 2\}$ for $i = 1, 2, 3$, and there is an edge $v — w$ if and only if $v_1 + v_2 + v_3 = w_1 + w_2 + w_3$.

Find how many connected components G has, and find also the chromatic number of the largest component.

3. Let G be a simple connected graph. Write down what is meant by

(i) a *Eulerian circuit* in G;

(ii) a *Hamiltonian circuit* in G.

Show that G has a Eulerian circuit if and only if every vertex of G has even valency.

Let $v_1, \ldots, v_n$ be the vertices of G, and suppose that $n \geq 3$. A new graph H is formed by adding n new vertices $w_1, \ldots, w_n$, and new edges as follows. There is an edge w_i — w_j precisely when there is an edge v_i — v_j in G, and there are also n edges v_i — w_i , $i = 1, \ldots, n$.

Decide, with reasoning, whether or not the following assertions are true. If an assertion is sometimes true, you should describe the circumstances in which it is true.

(a) If G has a Hamiltonian circuit, then H has a Hamiltonian circuit.

(b) If G has a Hamiltonian path that is not a circuit, then H has a Hamiltonian circuit.

(c) If G has a Eulerian circuit, then H has a Eulerian trail that is not a circuit.

(d) If G has a Eulerian trail that is not a circuit, H has either a Eulerian trail or circuit.

4. Define

(i) a *tree*;

(ii) a *spanning tree* for a graph G.

Describe the Breadth First Search method for constructing a spanning tree of a simple connected graph, and explain how it organises the vertices of the graph into layers.

Show that a simple connected graph is bipartite if and only if it contains no odd cycles.

Let G and H be the graphs given by the adjacency matrices below. Determine which (if any) are bipartite. If a graph is not bipartite, find its chromatic number.

$$G : \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad H : \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

5. Let G be a network with a given capacity function k, and let f be a feasible flow on G. Define the following terms.

(i) The *value* $\mathrm{Val}(f)$ of f.

(ii) A *cut* $C = (L, R)$.

(iii) The *capacity* $\mathrm{Kap}(C)$ of a cut C.

State the *Cut Conservation Lemma*, and explain how this result leads to the inequality

$$\mathrm{Val}(f) \leq \mathrm{Kap}(C)$$

for any feasible flow f and cut C.

Deduce that if we have a flow f^* and a cut C^* with $\mathrm{Val}(f^*) = \mathrm{Kap}(C^*)$, then f^* is a maximal flow and C^* is a minimal cut.

The figure below is a network, with edge directions indicated by the arrows, and capacities given in brackets. Starting with the zero flow, find a maximal flow in the network. [Give the flow augmenting path that you use in each step.]

Confirm that your flow is maximal by finding the capacity of an appropriate cut.