1. An investor wishes to invest in a portfolio of four classes of shares: S_1, S_2, S_3, S_4. Let the percentage of the investment put into class S_k be x_k ($k = 1, 2, 3, 4$).

The investor asks that the total percentage $x_1 + x_2$ of S_1 and S_2 shares does not exceed 30%. Further, the total percentage $x_2 + x_4$ of S_2 and S_4 shares should be at least 40%.

Subject to these constraints, the investor seeks to maximise the dividend returns. The returns are 7% for S_1, 6% for S_2, 3% for S_3, and 5% for S_4.

The problem is to decide how the investment should be allocated.

(a) Express the above as a linear programming problem (LP_A) giving the objective function, the equality constraints, and the inequality constraints.

(b) Construct the standard form linear programming problem (LP_B) using slack variables x_5 and x_6 in the inequalities.

Show that using the variables x_4, x_5 and x_6 in a starting vertex gives a degenerate vertex.

(c) Modify LP_B by adding extra variables as needed to create a linear programming problem (LP_C) which has a non-degenerate starting vertex (to be specified) based on the slack and extra variables.

(d) For LP_B write down a feasible basic point using the basic variables $x_B = \{x_2, x_3, x_4\}$.

(e) Use the simplex algorithm to show that this x_B is optimal. How should the investment be allocated? What is the maximum dividend return?

2. Let $S = \{x \in \Sigma(b) \mid x \geq 0\}$ where $\Sigma(b)$ is the set of solutions of linear equations $Ax = b$.

(a) (i) Prove that S is convex.

(ii) Prove that a basic point x of S is a vertex.

(b) Let G be a graph. Define the terms

(i) a Hamiltonian circuit;

(ii) an Eulerian circuit.

Let Q_n be the n-cube. Show that, for $n \geq 2$, Q_n has a Hamiltonian circuit. Determine the values of n for which Q_n has an Eulerian circuit.

Suppose that Q_n does not have an Eulerian circuit. Construct a graph G with one more vertex than Q_n so that G does have an Eulerian circuit.

3. Define the following terms.

(i) A *connected component* of a graph G.

(ii) A *simple* graph.

(iii) A *tree.*

(iv) A *spanning tree* for a graph G.

Describe the Breadth First Search method for finding a spanning tree of a simple graph G. Does this method always give a spanning tree?

Let G be a simple connected graph. Show that the following statements are equivalent.

(a) G is a tree.

(b) Removing any edge of G gives a disconnected graph.

Show also that in a tree there is a unique path between any two vertices.

Let G be the graph that has as vertices the integers $3, \ldots, 24$ inclusive, with an edge $x - y$ whenever x divides y or y divides x. Find a spanning tree for the connected component of 3, and find the connected components of G.

4. Define the following terms.

(i) the *valency* $\delta(v)$ of a vertex v of G;

(ii) the *chromatic number* $\chi(G)$ of G;

(iii) a *bipartite* graph.

Show that a graph is bipartite if and only if it contains no odd cycles.

Let K_n be the complete graph with n vertices,$n \geq 3$, and let H_n be the graph obtained by deleting the edge $1 - n$. Find $\chi(H_n)$.

Let K'_n be another copy of K_n, with vertices labelled $1', \ldots, n'$, and let L_n be formed from K_n and K'_n by adjoining an edge $i - i'$ for each $i = 1, \ldots, n$; thus L_n has vertices $1, \ldots, n, 1', \ldots, n'$, and the edges in L_n are those in K_n and K'_n together with the new edges given above. Find $\chi(L_n)$.

5. Let G be a network. Define the following terms.

(i) A *feasible flow* on G.

(ii) The Conservation Law satisfied by such a flow.

(iii) A *flow augmenting path*.

Suppose that we can find a flow augmenting path for a flow f. Show how to use the path to construct a new flow $\overline{f}$ with $\mathrm{Val}(\overline{f}) > \mathrm{Val}(f)$. (You need *not* show that $\overline{f}$ satisfies the Conservation Law.)

Let G be a bipartite graph, with disjoint sets of vertices W and B. Explain the *Matching Problem* and state *Hall's Theorem*.

Prove Hall's Theorem by constructing a suitable network.

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