Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2007

This paper is also taken for the relevant examination for the Associateship.

M2N1 NUMERICAL ANALYSIS

Date: ? 2007 Time: ?

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Statistical tables will not be available.

1. For $x,\, y \in \mathbb{R}$ with $x^2 + y^2 \neq 0$ and $p,\, q \in \{1, 2, \cdots, n\}$ with $p < q$, let $G_{p,q}(x, y) \in \mathbb{R}^{n\times n}$ be such that

$$
\begin{aligned}
[G_{p,q}(x,y)]_{ij} &= \quad \delta_{ij} \qquad i,\, j = 1 \to n\,, \\
\text{except that} \qquad [G_{p,q}(x,y)]_{pp} &= \quad [G_{p,q}(x,y)]_{qq} = \frac{x}{(x^2+y^2)^{\frac{1}{2}}} \\
\text{and} \qquad [G_{p,q}(x,y)]_{pq} &= -[G_{p,q}(x,y)]_{qp} = \frac{y}{(x^2+y^2)^{\frac{1}{2}}}\,.
\end{aligned}
$$

Write down the components of $G_{p,q}(x, y)\, \underline{r}$ for any $\underline{r} \in \mathbb{R}^n$.

If $r_p = x$ and $r_q = y$, what are the components of $G_{p,q}(x, y)\, \underline{r}$?

Show that $G_{p,q}(x, y)$ is orthogonal.

Let

$$\underline{a}_1 = \begin{pmatrix} 9 \\ 12 \\ 0 \end{pmatrix}, \quad \underline{a}_2 = \begin{pmatrix} -6 \\ -8 \\ 20 \end{pmatrix} \quad \text{and} \quad \underline{b} = \begin{pmatrix} 300 \\ 600 \\ 900 \end{pmatrix}.$$

Use matrices of the type $G_{p,q}(x, y) \in \mathbb{R}^{3\times 3}$ to find $x_1,\, x_2 \in \mathbb{R}$ that minimises

$$\|\underline{b} - \sum_{i=1}^{2} x_i\, \underline{a}_i\|\,,$$

where $\|\cdot\|$ is the standard norm on $\mathbb{R}^n$.

Check your solution by solving the corresponding normal equations.

Find all $\underline{d} \in \mathbb{R}^3$ such that

$$\min_{x_1,\, x_2 \in \mathbb{R}} \|\underline{d} - \sum_{i=1}^{2} x_i\, \underline{a}_i\| = \|\underline{d}\|\,.$$

2. Let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{R}^n$, and $\| \cdot \| = [\, \langle \cdot,\, \cdot \rangle \,]^{1/2}$ the associated norm. Prove the Cauchy-Schwartz inequality

$$|\langle \underline{a},\, \underline{b} \rangle| \leq \|\underline{a}\|\, \|\underline{b}\| \qquad \forall\, \underline{a},\, \underline{b} \in \mathbb{R}^n,$$

with equality if and only if $\underline{a}$ and $\underline{b}$ are linearly dependent.

Let $A = M^T M$, where $M \in \mathbb{R}^{n \times n}$ has linearly independent columns. Show that

(i) A is symmetric positive definite,

(ii) $A_{jj} > 0 \qquad j = 1 \to n,$

(iii) $|A_{jk}| < (\, A_{jj}\, A_{kk} \,)^{\frac{1}{2}} \qquad j \neq k, \qquad j,\, k = 1 \to n.$

Define the Cholesky factorization of a symmetric positive definite matrix.

Show that $|r\, s| \leq \frac{1}{2}\, [\varepsilon\, r^2 + \varepsilon^{-1} s^2]$ for all $r,\, s \in \mathbb{R}$ and for any $\varepsilon > 0$.

Use this result to show that

$$\begin{pmatrix} 4 & -6 & 2 \\ -6 & 34 & 2 \\ 2 & 2 & 11 \end{pmatrix}$$

is positive definite.

Compute its Cholesky factorization.

3. Write down the Lagrange and Newton forms of the polynomial $p_n(x)$ of degree $\leq n$, which interpolates the data $\{x_i,\ f(x_i)\}_{i=0}^{n}$, where the points $x_i \in [-1,1]$ are distinct.

Establish that the interpolating polynomial is unique.

Show that for any $x \neq x_j$, $j = 0 \to n$,

$$f(x) - p_n(x) = f[x_0, x_1, \cdots, x_n, x] \prod_{j=0}^{n}(x - x_j).$$

Show that if $f \in C^{n+1}[-1,1]$ then for all $x \in [-1,1]$ there exists a $\xi \in [-1,1]$, dependent on x, such that

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n}(x - x_i)\,.$$

If $\{x_i\}_{i=0}^{n}$ are chosen to be the $(n+1)$ zeroes of the Chebyshev polynomial

$$T_{n+1}(x) = \cos(\,(n+1)\,\cos^{-1} x\,) = 2^n\, x^{n+1} + \cdots\ ,$$

show that

$$\max_{-1\leq x\leq 1} |f(x) - p_n(x)| \leq \frac{2^{-n}}{(n+1)!} \max_{-1\leq x\leq 1} |f^{(n+1)}(x)|\,.$$

Approximation to e^{2x} on $[-1,1]$ is required to an absolute accuracy of 10^{-2}. With the above choice of interpolation points, what is minimum degree of $p_n(x)$ which is guaranteed to achieve this accuracy ?

If a continuous piecewise linear interpolating polynomial is used instead, with equally spaced nodes $y_j = -1 + \frac{2j}{J}$, $j = 0 \to J$, then what is the minimum value of J needed to achieve the required accuracy ?

[It is sufficient to note that $e^2 \approx 7.4$]

4. The Chebyshev polynomial of degree j, $j \geq 0$, is defined by

$$T_j(x) = \cos(j(\cos^{-1} x)) \qquad \forall\, x \in [-1, 1].$$

For $j \geq 1$, derive the recurrence relation

$$T_{j+1}(x) + T_{j-1}(x) = 2\, x\, T_j(x).$$

Show that $\{T_j(x)\}_{j\geq 0}$ is a set of orthogonal polynomials with respect to the inner product

$$\langle f,\, g\rangle = \int_{-1}^{1} (1 - x^2)^{-\frac{1}{2}}\, f(x)\, g(x)\, dx$$

on $C[-1, 1]$. In addition, show that

$$\|T_j\|^2 = \begin{cases} \frac{\pi}{2} & j \geq 1\,, \\ \pi & j = 0\,; \end{cases}$$

where $\|\cdot\| = [\langle\cdot, \cdot\rangle]^{\frac{1}{2}}$ is the associated norm on $C[-1, 1]$.

Given $f \in C[-1, 1]$, show that

$$p_n^*(x) = \sum_{j=0}^{n} \frac{\langle f, T_j\rangle}{\|T_j\|^2}\, T_j(x)$$

is such that

$$\langle f - p_n^*,\, p_n\rangle = 0 \qquad \forall\, p_n \in \mathbb{P}_n,\ \text{polynomials of degree } \leq n.$$

Hence show that

$$\|f - (p_n^* + p_n)\|^2 = \|f - p_n^*\|^2 + \|p_n\|^2 \qquad \forall\, p_n \in \mathbb{P}_n\,,$$

and so that p_n^* is best approximation from $\mathbb{P}_n$ to f with respect to $\|\cdot\|$.

In the case of $f(x) = (1 - x^2)^{\frac{1}{2}}\, x^2$, find p_0^*, p_1^* and p_2^*.

5. For all $f,\, g \in C[a,b]$ let

$$\langle f,\, g\rangle = \int_a^b w(x)\, f(x)\, g(x)\, dx\,,$$

where w is a positive weight function. Let $\phi_0(x) = 1$, $\phi_1(x) = x - \alpha_0$ and

$$\phi_{n+1}(x) = (x - \alpha_n)\,\phi_n(x) - \beta_n\,\phi_{n-1}(x), \qquad n \geq 1;$$

where

$$\alpha_n = \frac{\langle x\,\phi_n,\, \phi_n\rangle}{\langle \phi_n,\, \phi_n\rangle}, \quad n \geq 0, \qquad \text{and} \qquad \beta_n = \frac{\langle \phi_n,\, \phi_n\rangle}{\langle \phi_{n-1},\, \phi_{n-1}\rangle}, \quad n \geq 1\,.$$

Prove, by induction or otherwise, that $\{\phi_n(x)\}_{n\geq 0}$ is a set of orthogonal monic polynomials, $\phi_n \in \mathbb{P}_n$, with respect to $\langle \cdot,\, \cdot\rangle$.

Assuming that $\phi_{n+1}(x)$ has $n+1$ distinct zeros $\{x_i^*\}_{i=0}^n$ in $[a,b]$, show that on choosing

$$\omega_i^* = \int_a^b w(x) \prod_{j=0,\, j\neq i}^{n} \frac{(x - x_j^*)}{(x_i^* - x_j^*)}\, dx, \qquad i = 0 \to n,$$

then the quadrature formula

$$I_n^*(f) = \sum_{i=0}^{n} \omega_i^*\, f(x_i^*) \qquad \text{approximating} \qquad I(f) = \int_a^b w(x)\, f(x)\, dx$$

is exact for any $f \in \mathbb{P}_{2n+1}$.

For the case $[a,b] \equiv [0,1]$ and $w(x) = x^{-\frac{1}{4}}$ construct $I_0^*(f)$.