Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M2N1

Numerical Analysis

Date: Tuesday, 23rd May 2006 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (a) Use Givens rotations to compute the least squares solution $\underline{x}^\star$ of the overdetermined linear system $A\,\underline{x} = \underline{b}$, where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 4 \\ 1 & -3\sqrt{2} \end{pmatrix}, \qquad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \text{and} \qquad \underline{b} = \begin{pmatrix} 0 \\ 10 \\ 5\sqrt{2} \end{pmatrix}.$$

Calculate the error $\|A\,\underline{x}^\star - \underline{b}\|$, where $\|\underline{y}\| = (\underline{y}^T\,\underline{y})^{\frac{1}{2}}$.

Check your solution by solving the corresponding normal equations.

(b) For non-trivial $\underline{u}$, $\underline{v} \in \mathbb{R}^n$, let $B = I + \underline{u}\,\underline{v}^T \in \mathbb{R}^{n\times n}$, where $I \in \mathbb{R}^{n\times n}$ is the identity matrix. If $\underline{v}^T\underline{u} \neq -1$, verify that

$$B^{-1} = I - \frac{\underline{u}\,\underline{v}^T}{1 + \underline{v}^T\underline{u}}.$$

2. For a symmetric positive definite matrix $A \in \mathbb{R}^{n \times n}$, let

$$\langle \underline{x},\, \underline{y} \rangle_A = \underline{x}^T A\, \underline{y} \qquad \forall\ \underline{x},\, \underline{y} \in \mathbb{R}^n$$

be an inner product on $\mathbb{R}^n \times \mathbb{R}^n$, with $\| \cdot \|_A = [\, \langle \cdot,\, \cdot \rangle_A \,]^{\frac{1}{2}}$ being the associated norm on $\mathbb{R}^n$. Prove the Cauchy-Schwartz inequality

$$|\langle \underline{x},\, \underline{y} \rangle_A| \leq \|\underline{x}\|_A\, \|\underline{y}\|_A \qquad \forall\ \underline{x},\, \underline{y} \in \mathbb{R}^n,$$

with equality if and only if $\underline{x}$ and $\underline{y}$ are linearly dependent.

Let $\underline{e}_1 = (1,\, 0, \cdots, 0)^T \in \mathbb{R}^n$. Show that

$$A - \frac{(A\,\underline{e}_1)\,(A\,\underline{e}_1)^T}{\|\underline{e}_1\|_A^2} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & B & \\ 0 & & \end{pmatrix},$$

where $B \in \mathbb{R}^{(n-1)\times(n-1)}$ is such that $B^T = B$.

Let $\underline{v} = (0, \underline{u}^T)^T \in \mathbb{R}^n$, where $\underline{u} \in \mathbb{R}^{n-1}$. Show that

$$\underline{u}^T B\, \underline{u} = \|\underline{v}\|_A^2 - \frac{[\, \langle \underline{v}, \underline{e}_1 \rangle_A]^2}{\|\underline{e}_1\|_A^2}.$$

Hence show that B is positive definite.

Define the Cholesky factorization of A.

Show that $|r\, s| \leq \frac{1}{2}\,[r^2 + s^2] \quad \forall\ r,\, s \in \mathbb{R}$.

Use this result to show that

$$A = \begin{pmatrix} 36 & -18 & 6 \\ -18 & 34 & -13 \\ 6 & -13 & 21 \end{pmatrix}$$

is positive definite.

Compute its Cholesky factorization.

3. State the properties that a real-valued function $\langle\cdot,\,\cdot\rangle$ on $C[a,b]\times C[a,b]$ must satisfy for it to be an inner product.

Let $\|\cdot\|$ be the induced norm. Then for $j\geq 0$ let

$$\phi_j(x)=\frac{\psi_j(x)}{\|\psi_j\|}\,,$$

where $\psi_0(x)=1$ and for $j\geq 1$

$$\psi_j(x)=x^j-\sum_{i=0}^{j-1}\langle x^j,\phi_i\rangle\,\phi_i(x)\,.$$

Prove, using induction, that $\{\phi_j(x)\}_{j\geq 0}$ is a set of orthonormal polynomials with respect to $\langle\cdot,\,\cdot\rangle$.

Given $f\in C[a,b]$ and $\lambda_j\in\mathbb{R}$, $j=0\to n$, show that

$$\|f-\sum_{j=0}^{n}\lambda_j\,\phi_j\|^2=\|f\|^2+\sum_{j=0}^{n}[\,\lambda_j^2-2\,\lambda_j\,\langle f,\phi_j\rangle\,]\,.$$

Hence prove that

$$p_n^\star(x)=\sum_{j=0}^{n}\langle f,\phi_j\rangle\,\phi_j(x)$$

is the best approximation from $\mathbb{P}_n$, polynomials of degree $\leq n$, to f with respect to $\|\cdot\|$; i.e.

$$\|f-p_n^\star\|\leq\|f-p_n\|\qquad\forall\,p_n\in\mathbb{P}_n\,;$$

and that

$$\langle f-p_n^\star,p_n\rangle=0\qquad\forall\,p_n\in\mathbb{P}_n\,.$$

In the case $[a,b]\equiv[0,1]$ and

$$\langle f,\,g\rangle=\int_0^1 x^3\,f(x)\,g(x)\,dx\qquad\forall\,f,\,g\in C[0,1];$$

find $\{\phi_j(x)\}_{j=0}^{1}$, and hence the best approximation from $\mathbb{P}_1$ to x^2 with respect to $\|\cdot\|$.

4. Write down the Lagrange and Newton forms of the polynomial $p_n(z)$ of degree $\leq n$, which interpolates the data $\{z_j,\ f(z_j)\}_{j=0}^{n}$, where the points $z_j \in \mathbb{C}$ are distinct.

Prove that this interpolating polynomial is unique.

Establish the recurrence relation for divided differences

$$f[z_0, z_1, \cdots, z_j] = \frac{f[z_1, \cdots, z_j] - f[z_0, \cdots, z_{j-1}]}{z_j - z_0}, \qquad j = 1 \to n,$$

where $f[z_j] = f(z_j)$.

Discuss briefly the practical advantage of the Newton form.

Show that for any $z \neq z_j$, $j = 0 \to n$,

$$f(z) = p_n(z) + f[z_0, z_1, \cdots, z_n, z] \prod_{j=0}^{n} (z - z_j)\,.$$

Let

$$z_j = \omega^j\,, \quad j = 0 \to n\,, \qquad \text{where} \qquad \omega = e^{\frac{2\pi i}{n+1}} \in \mathbb{C}\,;$$

that is $\{z_j\}_{j=0}^{n}$ are the $(n+1)$th distinct roots of unity. Given that the corresponding Lagrange basis functions $\ell_j \in \mathbb{P}_n$, $j = 0 \to n$, are

$$\ell_0(z) = \frac{1}{n+1}\left(\frac{z^{n+1}-1}{z-1}\right) \equiv \frac{1}{n+1}\sum_{k=0}^{n} z^k \qquad \text{and} \qquad \ell_j(z) = \ell_0\left(\frac{z}{\omega^j}\right), \quad j = 1 \to n\,;$$

show that

$$p_n(z) = \frac{z^{n+1}-1}{n+1} \sum_{j=0}^{n} \frac{f(\omega^j)\,\omega^j}{z - \omega^j}\,.$$

5. Let $f \in C[a,b]$. Show that if a $p_n^\star \in \mathbb{P}_n$, polynomials of degree $\leq n$, satisfies

$$f(x_j) - p_n^\star(x_j) = (-1)^j\,\sigma\,E$$

at $(n+2)$ distinct points $a \leq x_0 < x_1 < \cdots < x_n < x_{n+1} \leq b$, where

$$E = \|f - p_n^\star\|_\infty = \max_{a \leq x \leq b} |f(x) - p_n^\star(x)|$$

and $\sigma = 1$ or -1, then

$$\|f - p_n^\star\|_\infty \leq \|f - p_n\|_\infty \qquad \forall\, p_n \in \mathbb{P}_n\,.$$

Let $q_n \in \mathbb{P}_n$ be such that

$$f(y_j) - q_n(y_j) = (-1)^j\,\xi_j\,,$$

where ξ_j has the same sign at each of the $(n+2)$ distinct points

$$a \leq y_0 < y_1 < \cdots < y_n < y_{n+1} \leq b\,.$$

By considering the sign of $p_n^\star - q_n$ at $\{y_j\}_{j=0}^{n+1}$, show that

$$\min_{j=0\to n+1} |\xi_j| \leq E\,.$$

By considering $q_1(x) = x - c$, where $c > 0$, and $y_0 = 0$, $y_1 = \frac{1}{2}$ and $y_2 = 1$; deduce that the best approximation $p_1^\star$ from $\mathbb{P}_1$ to $f(x) = x^3$ on $[a,b] \supseteq [0,1]$ satisfies

$$\|f - p_1^\star\|_\infty \geq \frac{3}{16}\,.$$