Imperial College London

UNIVERSITY OF LONDON

Course:	M2N1
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BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2005

This paper is also taken for the relevant examination for the Associateship.

M2N1 NUMERICAL ANALYSIS

Date: ? 2005 Time: ?

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Statistical tables will not be available.

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1. Use Givens rotations to compute the least squares solution $\underline{x}^\star$ of the overdetermined linear system $A\,\underline{x} = \underline{b}$, where

$$A = \begin{pmatrix} 4 & 20 \\ 0 & 4 \\ 0 & 3 \\ -3 & 0 \end{pmatrix}, \qquad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \text{and} \qquad \underline{b} = \begin{pmatrix} -3 \\ 8 \\ 1 \\ -19 \end{pmatrix}.$$

Calculate the error $\|A\,\underline{x}^\star - \underline{b}\|$.

Check your solution by solving the corresponding normal equations.

Find a $\underline{d} \in \mathbb{R}^4$, $\underline{d} \neq \underline{0}$, such that the least squares solution of the overdetermined linear system $A\,\underline{y} = \underline{d}$ is $\underline{y}^\star = \underline{0}$.

2. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix. Show that

$$\langle \underline{u},\, \underline{v}\rangle_A = \underline{u}^T A\,\underline{v} \qquad \forall\, \underline{u},\, \underline{v} \in \mathbb{R}^n$$

is an inner product on $\mathbb{R}^n$.

Assuming the Gram-Schmidt algorithm, show that there exists an invertible $P \in \mathbb{R}^{n\times n}$ such that $A = P^T P$.

Define the Cholesky factorization of A.

Show that $|r\,s| \leq \frac{1}{2}\,[r^2 + s^2] \quad \forall\, r,\, s \in \mathbb{R}$.

Use this result to show that

$$A = \begin{pmatrix} 9 & 3 & -3 \\ 3 & 5 & 1 \\ -3 & 1 & 11 \end{pmatrix}$$

is positive definite.

Compute its Cholesky factorization.

Use this factorization to find A^{-1}.

3. Let V be a real vector space. State the properties that a real-valued function $\langle \cdot, \cdot \rangle$ on $V \times V$ must satisfy for it to be an inner product.

Let U be a subspace of V with basis $\{\phi_i\}_{i=1}^n$. Given any $v \in V$, let

$$E(\underline{\lambda}) = \|v - \sum_{i=1}^{n} \lambda_i \, \phi_i\|^2, \qquad \text{where} \qquad \underline{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_n)^T \in \mathbb{R}^n$$

and $\| \cdot \| = [\langle \cdot, \cdot \rangle]^{\frac{1}{2}}$. Show that

$$E(\underline{\lambda}) = \|v\|^2 - 2\, \underline{\lambda}^T \underline{\mu} + \underline{\lambda}^T G\, \underline{\lambda},$$

where $\underline{\mu} \in \mathbb{R}^n$ and $G \in \mathbb{R}^{n \times n}$ are such that

$$\mu_i = \langle v, \phi_i \rangle \qquad \text{and} \qquad G_{ij} = \langle \phi_i, \phi_j \rangle, \qquad i,\, j = 1 \to n.$$

Derive the following results.

(a) G is a symmetric positive definite matrix.

(b) There exists a unique $\underline{\lambda}^\star \in \mathbb{R}^n$ such that $G\, \underline{\lambda}^\star = \underline{\mu}$.

(c) $E(\underline{\lambda}^\star + \underline{h}) = E(\underline{\lambda}^\star) + \underline{h}^T G\, \underline{h} \geq E(\underline{\lambda}^\star) \quad \forall\, \underline{h} \in \mathbb{R}^n$.

(d) $u^\star = \displaystyle\sum_{i=1}^{n} \lambda_i^\star \, \phi_i \in U$ is such that $\langle v - u^\star, u \rangle = 0 \quad \forall\, u \in U$.

(e) $E(\underline{\lambda}^\star) = \langle v - u^\star, v \rangle$.

Let $V = C[0,1]$ with inner product

$$\langle f, g \rangle = \int_0^1 x\, f(x)\, g(x)\, dx \quad \forall\, f,\, g \in C[0,1].$$

Let $U = \mathbb{P}_1$, polynomials of degree ≤ 1, with basis $\{1, x\}$.

If $v = 20\, x^2$, find the corresponding $u^\star$ and calculate $E(\underline{\lambda}^\star)$.

4. Write down the Lagrange form of $p_n \in \mathbb{P}_n$, which interpolates the data $\{x_i, f(x_i)\}_{i=0}^n$, where the points $\{x_i\}_{i=0}^n$ are distinct.

Establish the following results.

(a) This interpolating polynomial is unique.

(b) $\displaystyle\sum_{i=0}^{n}\left[\prod_{j=0,\, j\neq i}^{n}\left(\frac{x-x_j}{x_i-x_j}\right)\right] = 1 \qquad \forall\, x \in \mathbb{R}.$

Write down the Newton form of p_n and establish the following results.

(c) $\displaystyle f[x_0, x_1, \cdots, x_n] = \sum_{i=0}^{n}\left[f(x_i)\prod_{j=0,\, j\neq i}^{n}\frac{1}{(x_i-x_j)}\right].$

(d) For any $x \neq x_i, i = 0 \to n$,

$$f(x) - p_n(x) = f[x_0, x_1, \cdots, x_n, x]\prod_{j=0}^{n}(x - x_j).$$

If $f(x) = (\alpha - x)^{-1}$ with $\alpha \neq x_i, i = 0 \to n$, use the results above to show the following.

(e) $\displaystyle f[x_0, x_1, \cdots, x_n] = \prod_{j=0}^{n}\frac{1}{(\alpha - x_j)}.$

(f) For any $x \neq \alpha$ or $x_i, i = 0 \to n$,

$$f(x) - p_n(x) = \frac{1}{(\alpha - x)}\prod_{j=0}^{n}\left(\frac{x - x_j}{\alpha - x_j}\right).$$

5. For all $f,\, g \in C[-L, L]$ let

$$\langle f,\, g\rangle = \int_{-L}^{L} w(x)\, f(x)\, g(x)\, dx\,,$$

where $w \in C[-L, L]$ is a positive <u>even</u> weight function. Let $\phi_0(x) = 1$, $\phi_1(x) = x$ and for $n \geq 1$

$$\phi_{n+1}(x) = x\,\phi_n(x) - \beta_n\,\phi_{n-1}(x), \qquad \text{where} \qquad \beta_n = \frac{\langle \phi_n,\, \phi_n\rangle}{\langle \phi_{n-1},\, \phi_{n-1}\rangle}.$$

Prove by induction that $\{\phi_n(x)\}_{n\geq 0}$ is a set of orthogonal monic polynomials, $\phi_n \in \mathbb{P}_n$, with respect to $\langle \cdot,\, \cdot\rangle$ such that ϕ_n is an even (odd) function if n is even (odd).

Assuming that $\phi_{n+1}(x)$ has $n+1$ distinct zeros $\{x_i^\star\}_{i=0}^{n} \in [-L, L]$, show that on choosing

$$\omega_i^\star = \int_{-L}^{L} w(x) \left[\prod_{j=0,\, j\neq i}^{n} \left(\frac{x - x_j^\star}{x_i^\star - x_j^\star}\right)\right] dx, \qquad i = 0 \to n,$$

then the quadrature formula

$$I_n^\star(f) = \sum_{i=0}^{n} \omega_i^\star\, f(x_i^\star) \qquad \text{approximating} \qquad I(f) = \int_{-L}^{L} w(x)\, f(x)\, dx$$

is exact for any $f \in \mathbb{P}_{2n+1}$.

For the case $L = 1$ and $w(x) = 3 + 4\,|x|$ construct the sampling points and the weights for $I_0^\star(f)$ and $I_1^\star(f)$.