UNIVERSITY OF LONDON

Course:	M2N1
Setter:	Barrett
Checker:	G. Moore
Editor:	G. Moore
External:	Cowley
Date:	April 5, 2004

BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2004

This paper is also taken for the relevant examination for the Associateship.

M2N1 NUMERICAL ANALYSIS

Date: Friday, 21st May 2004 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Statistical tables will not be available.

Setter's signature	Checker's signature	Editor's signature
....	....	....

1. For any non-singular $A \in \mathbb{R}^{n\times n}$ and non-trivial $\underline{u}$, $\underline{v} \in \mathbb{R}^n$, let $B = (A + \underline{u}\,\underline{v}^T)$.
If $\underline{v}^T A^{-1}\,\underline{u} \neq -1$, verify that

$$B^{-1} = A^{-1} - \frac{A^{-1}\,\underline{u}\,\underline{v}^T A^{-1}}{1 + \underline{v}^T A^{-1}\,\underline{u}}\,.$$

Find a $\underline{u} \in \mathbb{R}^3$ such that

$$C = \begin{pmatrix} 4 & -2 & 6 \\ -2 & 10 & -12 \\ 6 & -12 & 54 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 18 \end{pmatrix} + \underline{u}\,\underline{u}^T\,.$$

Hence find C^{-1}.

Using the above decomposition of C, show that C is positive definite.

Find a lower triangular $L \in \mathbb{R}^{3\times 3}$ with strictly positive diagonal elements such that $C = L\,L^T$.

Find L^{-1}.

Write down the general formula for C^{-1} in terms of L^{-1}.

2. For $x,\, y \in \mathbb{R}$ with $x^2 + y^2 \neq 0$ and $p,\, q \in \{1, 2, \cdots, n\}$ with $p < q$, let $G_{p,q}(x, y) \in \mathbb{R}^{n \times n}$ be such that

$$
\begin{aligned}
[G_{p,q}(x,y)]_{ij} &= \quad \delta_{ij} \qquad i,\, j = 1 \to n\,, \\
\text{except that} \quad [G_{p,q}(x,y)]_{pp} &= \quad [G_{p,q}(x,y)]_{qq} = \frac{x}{(x^2+y^2)^{\frac{1}{2}}} \\
\text{and} \quad [G_{p,q}(x,y)]_{pq} &= -[G_{p,q}(x,y)]_{qp} = \frac{y}{(x^2+y^2)^{\frac{1}{2}}}\,.
\end{aligned}
$$

Write down the components of $G_{p,q}(x, y)\, \underline{r}$ for any $\underline{r} \in \mathbb{R}^n$.

If $r_p = x$ and $r_q = y$, what are these components ?

Show that $G_{p,q}(x, y)$ is orthogonal.

Let

$$
\underline{a}_1 = \begin{pmatrix} 3 \\ 0 \\ 4 \\ 0 \end{pmatrix}, \quad \underline{a}_2 = \begin{pmatrix} 5 \\ 4 \\ 0 \\ 2\sqrt{2} \end{pmatrix} \quad \text{and} \quad \underline{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 50\sqrt{2} \end{pmatrix}.
$$

Use matrices of the type $G_{p,q}(x, y) \in \mathbb{R}^{4\times 4}$ to find $x_1,\, x_2 \in \mathbb{R}$ that minimises

$$
\|\underline{b} - \sum_{i=1}^{2} x_i\, \underline{a}_i\|\,.
$$

Check your solution by solving the corresponding normal equations.

3. Write down the Lagrange form of $p_n \in \mathbb{P}_n$, which interpolates the data $\{x_i, f_i\}_{i=0}^n$, where the points $\{x_i\}_{i=0}^n$ are distinct.

Establish that this interpolating polynomial is unique.

Let $\{y_j\}_{j=0}^m$ be another set of distinct points: thus the $(n+1)\,(m+1)$ distinct points in the plane,

$$(x_i, y_j) \quad i = 0 \to n,\ j = 0 \to m,$$

form a rectangular array aligned with the axes. Let $\mathbb{P}_{n,m}$ be the space of polynomials in two variables of degree at most n in the first variable and of degree at most m in the second variable, i.e. $\mathbb{P}_{n,m}$ consists of polynomials

$$\sum_{i=0}^{n} \sum_{j=0}^{m} a_{i,j}\, x^i\, y^j$$

for arbitrary constants $a_{i,j}$ $i = 0 \to n$, $j = 0 \to m$.

For $i = 0 \to n$, $j = 0 \to m$ find $\ell_{i,j}(x, y) \in \mathbb{P}_{n,m}$ such that

$$\ell_{i,j}(x_i, y_j) = 1$$

and at the other $(n+1)\,(m+1) - 1$ points (x_k, y_l)

$$\ell_{i,j}(x_k, y_l) = 0\,.$$

Given data values $z_{i,j} \in \mathbb{R}$, $i = 0 \to n$, $j = 0 \to m$; prove that there exists a $p_{n,m} \in \mathbb{P}_{n,m}$ such that

$$p_{n,m}(x_i, y_j) = z_{i,j} \quad i = 0 \to n,\ j = 0 \to m.$$

By using the fact that any element of $\mathbb{P}_{n,m}$ can be written in the form

$$\sum_{k=0}^{n} b_k(y)\, x^k$$

where each $b_k \in \mathbb{P}_m$, prove that the interpolating $p_{n,m}$ above is unique.

Find $p_{1,1} \in \mathbb{P}_{1,1}$ that interpolates a continuous function $g(x, y)$ at the points $(-1, -1)$, $(-1, 1)$, $(1, -1)$ and $(1, 1)$.

Is this interpolation problem guaranteed to be well-posed if we choose any four distinct points in the plane, not necessarily in a rectangular array aligned with the axes ?

Is it well-posed for the the points $(-1, 0)$, $(0, -1)$, $(0, 1)$ and $(1, 0)$?

4. Let $f \in C[a,b]$. Show that if a $p_n^\star \in \mathbb{P}_n$, polynomials of degree $\leq n$, satisfies

$$f(x_j) - p_n^\star(x_j) = (-1)^j \, \sigma \, E$$

at $(n+2)$ distinct points $a \leq x_0 < x_1 < \cdots x_n < x_{n+1} \leq b$, where

$$E = \|f - p_n^\star\|_\infty = \max_{a \leq x \leq b} |f(x) - p_n^\star(x)|$$

and $\sigma = 1$ or -1, then

$$\|f - p_n^\star\|_\infty \leq \|f - p_n\|_\infty \qquad \forall \; p_n \in \mathbb{P}_n \, .$$

State the Chebyshev equioscillation theorem.

Let $[a,b] \equiv [-d,d]$, f be an even function, and $p_n^\star$ be the best approximation to f from $\mathbb{P}_n$ in $\|\cdot\|_\infty$ on $[-d,d]$. By considering $f(x) - p_n^\star(-x)$, use the equioscillation theorem to deduce that $p_n^\star$ is an even function.

Hence find $p_1^\star$ for $f(x) = (4 - x^2)^{\frac{1}{2}}$ on $[-2,2]$.

Compare this to the best approximation to f from $\mathbb{P}_1$ with respect to the norm

$$\|g\| = \left(\int_{-2}^{2} [g(x)]^2 \, dx \right)^{\frac{1}{2}} .$$

5. State the properties that a real-valued function $\langle \cdot, \cdot \rangle$ on $C[a,b] \times C[a,b]$ must satisfy for it to be an inner product.

Prove the Cauchy-Schwartz inequality

$$|\langle f, g \rangle| \leq \|f\| \, \|g\| \qquad \forall \, f, \, g \in C[a,b],$$

where $\| \cdot \| = [\, \langle \cdot, \cdot \rangle \,]^{1/2}$ is the associated norm on $C[a,b]$.

The Chebyshev polynomial of degree n, $n \geq 0$, is defined by

$$T_n(x) = \cos(n(\cos^{-1} x)) \qquad \forall \, x \in [-1, 1].$$

For $n \geq 1$, derive the recurrence relation

$$T_{n+1}(x) + T_{n-1}(x) = 2 \, x \, T_n(x) \, .$$

Show that $\{T_n(x)\}_{n \geq 0}$ is a set of orthogonal polynomials with respect to the inner product

$$\langle f, g \rangle = \int_{-1}^{1} (1 - x^2)^{-\frac{1}{2}} \, f(x) \, g(x) \, dx.$$

Let

$$I(f) = \int_{-1}^{1} (1 - x^2)^{-\frac{1}{2}} \, f(x) \, dx \, .$$

Find the sampling points $\{x_i\}_{i=0}^{2} \in [-1, 1]$ and the weights $\{w_i\}_{i=0}^{2}$ so that

$$I_1(f) = w_0 \, f(x_0) \qquad \text{and} \qquad I_3(f) = w_1 \, f(x_1) + w_2 \, f(x_2)$$

are such that $I_n(f) = I(f) \quad \forall \, f \in \mathbb{P}_n \, .$