UNIVERSITY OF LONDON

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

BSc/MSci EXAMINATION(MATHEMATICS) MAY-JUNE 2003

This paper is also taken for the relevant examination for the Associateship

M2N1 NUMERICAL ANALYSIS

DATE : ? TIME : ?

Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. **(a)** Use Givens rotations to compute the least squares solution of the overdetermined linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 3 & 2 \\ 0 & 1 \\ 4 & 1 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}.$$

Calculate the error $\|A\mathbf{x} - \mathbf{b}\|$.

Check your solution by solving the corresponding normal equations.

(b) Let $\mathbf{u} \in \mathbb{R}^n$ with $\|\mathbf{u}\| = 1$. Define

$$P = I - 2\,\mathbf{u}\,\mathbf{u}^T \in \mathbb{R}^{n\times n}.$$

Prove that P is a symmetric orthogonal matrix such that

$$\begin{array}{lll} & \text{(i)} & P\,\mathbf{u} = -\mathbf{u}\,, \\ \text{and} & \text{(ii)} & P\,\mathbf{v} = \mathbf{v} \qquad \forall\, \mathbf{v} \in \mathbb{R}^n \quad \text{with} \quad \mathbf{v}^T\mathbf{u} = 0\,. \end{array}$$

Find $\displaystyle\sum_{i=1}^{n} P_{ii}\,.$

2. State the properties that a real-valued function $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n \times \mathbb{R}^n$ must satisfy for it to be an inner product.

Let $\|\cdot\| = [\, \langle \cdot, \cdot \rangle \,]^{1/2}$ be the associated norm on $\mathbb{R}^n$. Prove the Cauchy-Schwartz inequality

$$|\langle \mathbf{a}, \mathbf{b} \rangle| \leq \|\mathbf{a}\| \, \|\mathbf{b}\| \qquad \forall \, \mathbf{a}, \mathbf{b} \in \mathbb{R}^n,$$

with equality if and only if $\mathbf{a}$ and $\mathbf{b}$ are linearly dependent.

Let $A = M^T M$, where $M \in \mathbb{R}^{n \times n}$ has linearly independent columns. Show that

(i) A is symmetric positive definite,

(ii) $A_{jj} > 0 \qquad j = 1 \to n,$

(iii) $|A_{jk}| < (\, A_{jj} \, A_{kk} \,)^{\frac{1}{2}} \qquad j \neq k, \qquad j, k = 1 \to n.$

Define the Cholesky factorization of a symmetric positive definite matrix.

Assuming that

$$\begin{bmatrix} 4 & 2 & 6 \\ 2 & 10 & 0 \\ 6 & 0 & 35 \end{bmatrix}$$

is positive definite, compute its Cholesky factorization.

3. Let $x_0 < x_1 < \cdots < x_{n-1} < x_n$ and $f \in C[x_0, x_n]$. Write down the Newton form of the polynomial $p_n(x)$ of degree $\leq n$, which interpolates the data $\{x_i,\, f(x_i)\}_{i=0}^n$.

Establish the recurrence relation for divided differences

$$f[x_0, x_1, \cdots, x_j] = \frac{f[x_1, \cdots, x_j] - f[x_0, \cdots, x_{j-1}]}{x_j - x_0} \qquad j = 1 \to n,$$

where $f[x_j] = f(x_j)$.

Show that for any $x \neq x_j$, $j = 0 \to n$,

$$f(x) = p_n(x) + f[x_0, x_1, \cdots, x_n, x] \prod_{j=0}^{n} (x - x_j)\,.$$

If $f \in C^n(\mathbb{R})$, show that

$$f[x_0, x_1, \cdots, x_j] = \frac{f^{(j)}(\xi_j)}{j!}, \qquad j = 1 \to n,$$

where $\xi_j \in (x_0, x_j)$.

For the case $f(x) = e^{-x}$, show that

$$(-1)^j\, f[x_0, x_1, \cdots, x_j] \geq 0, \qquad j = 0 \to n,$$

and

$$0 \leq p_n(x) \leq f(x) \qquad \forall\; x < x_0\,.$$

4. Let $f \in C[a,b]$. Show that if a $p_n^\star \in \mathbb{P}_n$, polynomials of degree $\leq n$, satisfies

$$f(x_j) - p_n^\star(x_j) = (-1)^j \,\sigma\, E$$

at $(n+2)$ distinct points $a \leq x_0 < x_1 < \cdots x_n < x_{n+1} \leq b$, where

$$E = \|f - p_n^\star\|_\infty = \max_{a \leq x \leq b} |f(x) - p_n^\star(x)|$$

and $\sigma = 1$ or -1, then

$$\|f - p_n^\star\|_\infty \leq \|f - p_n\|_\infty \qquad \forall\, p_n \in \mathbb{P}_n\,.$$

The Chebyshev polynomial of degree n, $n \geq 0$, is defined by

$$T_n(x) = \cos(n(\cos^{-1} x)) \qquad \forall\, x \in [-1,1].$$

For $n \geq 1$, derive the recurrence relation

$$T_{n+1}(x) + T_{n-1}(x) = 2\,x\,T_n(x)\,.$$

For $n \geq 1$, prove that the coefficient of x^n in $T_n(x)$ is 2^{n-1}.

Let $[a,b] \equiv [-1,1]$. Show that the best approximation to x^{n+1} by $\mathbb{P}_n$ in $\|\cdot\|_\infty$ is $p_n^\star(x) = x^{n+1} - 2^{-n}\,T_{n+1}(x)$.

Hence calculate explicitly the best approximation to x^3 by $\mathbb{P}_2$ and the corresponding $\|\cdot\|_\infty$ error.

5. For all $f,\, g \in C[a,b]$ let

$$\langle f,\, g \rangle = \int_a^b w(x)\, f(x)\, g(x)\, dx\,,$$

where w is a positive weight function. Let $\phi_0(x) = 1$, $\phi_1(x) = x - a_0$ and

$$\phi_{n+1}(x) = (x - a_n)\,\phi_n(x) - b_n\,\phi_{n-1}(x), \qquad n \geq 1;$$

where

$$a_n = \frac{\langle x\phi_n,\, \phi_n \rangle}{\langle \phi_n,\, \phi_n \rangle}, \quad n \geq 0, \qquad \text{and} \qquad b_n = \frac{\langle \phi_n,\, \phi_n \rangle}{\langle \phi_{n-1},\, \phi_{n-1} \rangle}, \quad n \geq 1\,.$$

Prove that $\{\phi_n(x)\}_{n \geq 0}$ is a set of orthogonal monic polynomials, $\phi_n \in \mathbb{P}_n$, with respect to $\langle \cdot,\, \cdot \rangle$.

Assuming that $\phi_{n+1}(x)$ has $n+1$ distinct zeros $\{x_i\}_{i=0}^n$, show that on choosing

$$\omega_i = \int_a^b w(x) \prod_{j=0,\, j \neq i}^{n} \frac{(x - x_j)}{(x_i - x_j)}\, dx, \qquad i = 0 \to n,$$

then the quadrature formula

$$\sum_{i=0}^{n} \omega_i\, f(x_i) \qquad \text{approximating} \qquad \int_a^b w(x)\, f(x)\, dx$$

is exact for any $f \in \mathbb{P}_{2n+1}$.

For the case $[a,b] \equiv [0,1]$ and $w(x) = x^{-\frac{1}{2}}$ construct a one point quadrature formula, which is exact for any $f \in \mathbb{P}_1$.