UNIVERSITY OF LONDON

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

BSc/MSci EXAMINATION(MATHEMATICS) MAY-JUNE 2002

This paper is also taken for the relevant examination for the Associateship

M2N1 NUMERICAL ANALYSIS

DATE : Monday 20 May 2002 TIME : 2.00pm – 4.00pm

Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Use Given's rotations to compute the least squares solution of the overdetermined linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & -8 \\ 2 & -1 \\ 2 & 14 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} 4 \\ 35 \\ -1 \end{bmatrix}.$$

Calculate the error $\|A\mathbf{x} - \mathbf{b}\|$.

Check your solution by solving the corresponding normal equations.

2. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix.

Show that

$$\langle \mathbf{u}, \mathbf{v}\rangle_A = \mathbf{u}^T A\,\mathbf{v} \qquad \forall\ \mathbf{u}, \mathbf{v} \in \mathbb{R}^n$$

is an inner product on $\mathbb{R}^n$.

Assuming the Gram-Schmidt algorithm, show that A has a Cholesky factorization; that is, there exists a lower triangular matrix $L \in \mathbb{R}^{n\times n}$ with strictly positive diagonal elements such that $A = L\,L^T$.

Assuming that

$$\begin{bmatrix} 4 & -10 & 2 \\ -10 & 34 & -17 \\ 2 & -17 & 18 \end{bmatrix}$$

is positive definite, compute its Cholesky factorization.

Use this factorization, to solve the linear system

$$\begin{bmatrix} 4 & -10 & 2 \\ -10 & 34 & -17 \\ 2 & -17 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -10 \\ 7 \\ 22 \end{bmatrix}.$$

3. State the properties that a real-valued function $\langle \cdot, \cdot \rangle$ on $C[a,b] \times C[a,b]$ must satisfy for it to be an inner product.

Prove the Cauchy-Schwartz inequality

$$|\langle f, g \rangle| \leq \|f\| \, \|g\| \qquad \forall \, f, \, g \in C[a,b],$$

where $\| \cdot \| = [\, \langle \cdot, \cdot \rangle \,]^{1/2}$ is the associated norm on $C[a,b]$.

For $j \geq 0$ let

$$\phi_j(x) = \psi_j(x) \, / \, \|\psi_j\| \, ,$$

where $\psi_0(x) = 1$ and for $j \geq 1$

$$\psi_j(x) = x^j - \sum_{i=0}^{j-1} \langle x^j, \, \phi_i \rangle \, \phi_i(x) \, .$$

Prove, using induction, that $\{\phi_j(x)\}_{j \geq 0}$ is a set of orthonormal polynomials with respect to $\langle \cdot, \cdot \rangle$.

Given $f \in C[a,b]$, prove that

$$p_n^\star(x) = \sum_{j=0}^{n} \langle f, \, \phi_j \rangle \, \phi_j(x)$$

is the best approximation from $\mathbb{P}_n$, polynomials of degree $\leq n$, to f with respect to $\| \cdot \|$; i.e.

$$\|f - p_n^\star\| \leq \|f - p_n\| \qquad \forall \, p_n \in \mathbb{P}_n \, .$$

In the case $[a,b] \equiv [0,1]$ and

$$\langle f, \, g \rangle = \int_0^1 f(x) \, g(x) \, dx \qquad \forall \, f, \, g \in C[0,1];$$

find the best approximation from $\mathbb{P}_1$ to x^2 with respect to $\| \cdot \|$.

4. Write down the Lagrange and Newton forms of the polynomial $p_n(x)$ of degree $\leq n$, which interpolates the data $\{x_i,\ f(x_i)\}_{i=0}^n$, where the points $x_i \in [-1, 1]$ are distinct. Discuss briefly the advantage of the Newton form.

Establish that the interpolating polynomial is unique and that if $f \in C^{n+1}[-1, 1]$ then for all $x \in [-1, 1]$ there exists a $\xi \in [-1, 1]$, dependent on x, such that

$$f(x) = p_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)\,.$$

Let $\{x_i\}_{i=0}^n$ be the $(n+1)$ zeroes of the Chebyshev polynomial

$$T_{n+1}(x) = \cos(\,(n+1)\ \cos^{-1} x\,) = 2^n\, x^{n+1} + \cdots\ .$$

Show that

$$\max_{-1 \leq x \leq 1} |f(x) - p_n(x)| \leq \frac{2^{-n}}{(n+1)!} \max_{-1 \leq x \leq 1} |f^{(n+1)}(x)|\,.$$

Approximation to $e^{2\,x}$ on $[-1, 1]$ is required to an absolute accuracy of 10^{-2}. With the above choice of interpolation points, what is minimum degree of $p_n(x)$ which is guaranteed to achieve this accuracy ?

[Note that $e^2 \leq 7.4$]

5. Let $f \in C[a,b]$. Show that if a $p_n^\star \in \mathbb{P}_n$, polynomials of degree $\leq n$, satisfies

$$f(x_j) - p_n^\star(x_j) = (-1)^j \, \sigma \, E$$

at $(n+2)$ distinct points $a \leq x_0 < x_1 < \cdots x_n < x_{n+1} \leq b$, where

$$E = \|f - p_n^\star\|_\infty = \max_{a \leq x \leq b} |f(x) - p_n^\star(x)|$$

and $\sigma = 1$ or -1, then

$$\|f - p_n^\star\|_\infty \leq \|f - p_n\|_\infty \qquad \forall \; p_n \in \mathbb{P}_n \, .$$

Let $q_n \in \mathbb{P}_n$ be such that

$$f(y_j) - q_n(y_j) = (-1)^j \, \xi_j \, ,$$

where ξ_j has the same sign at each of the $(n+2)$ distinct points

$$a \leq y_0 < y_1 < \cdots y_n < y_{n+1} \leq b \, .$$

By considering the sign of $q_n - p_n^\star$ at $\{y_j\}_{j=0}^{n+1}$ show that

$$\min_{j=0,\,1,\cdots,n+1} |\xi_j| \leq E \, .$$

By considering $q_1(x) = x + 0.1$, deduce that the best approximation $p_1^\star$ to $f(x) = x^{\frac{1}{2}}$ on $[0,1]$ satisfies

$$\|f - p_1^\star\|_\infty \geq 0.1 \, .$$