1. Consider the ODE

$$Ly \equiv x^2y'' - 2xy' + (2 + x^2)y = x^3 \tag{1}$$

(i) Show that $y = x\cos x$ is a solution to the homogenous equation $Ly = 0$.

(ii) Evaluate the Wronskian for (1), and hence find a second solution to the equation $Ly = 0$.

(iii) Construct a Green's function for (1) appropriate for an *initial* value problem $y(1) = \alpha$ and $y'(1) = \beta$, and hence find the solution for the case $\alpha = \beta = 0$.

2. Find the general series solution about $x = 0$ for the equation

$$(1 - x^2)y'' - 3xy' + \lambda y = 0$$

where λ is a real constant.

(i) Show that the solution has two parts, one even and one odd in x.

(ii) What is the radius of convergence for either series?

(iii) Determine the eigenvalues λ for which one or other series terminates as a polynomial. Call these $y_n(x)$, indicating a polynomial degree n, and write down the form for $y_0(x)$, $y_1(x)$, $y_2(x)$ and $y_3(x)$, in each case to within an arbitrary constant.

(iv) Rewrite the equation in Sturm-Liouville form and hence deduce the orthogonality relation between the eigenfunctions.

3. The Fourier Transform of the function $f(x)$ is defined by

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx.$$

State the inverse transform which expresses $f(x)$ in terms of $\hat{f}(k)$. The function $f(x)$ is defined by

$$f(x) = \begin{matrix} e^{-x}, & x > 0 \\ 0, & x < 0 \end{matrix} \quad .$$

(i) Find $\hat{f}(k)$.

(ii) Use the convolution theorem to find the function $p(x)$ which satisfies the integral equation

$$\int_0^{\infty} f(y)p(x - y)dy = x^2 f(x).$$

4. The function $f(x)$ is defined as follows

$$\begin{aligned} f(x) &= x^2, \quad -\pi < x < \pi \\ f(x+2\pi) &= f(x). \end{aligned}$$

Show that, in -$\pi < x < \pi$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx.$$

By differentiating and integration this series, infer the Fourier series of

(i) $x, \quad -\pi < x < \pi$

(ii) $x^3, \quad -\pi < x < \pi.$

Finally, by considering a suitable value for x in the Fourier Series for x^2, show that

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

5. The function $u(r, \theta, t)$ satisfies the diffusion equation

$$\nabla^2 u = \partial u / \partial t$$

within the disc $r < a$ for time $t > 0$. It is nonsingular at the origin and takes the value

$$u(a, \theta, t) = 0$$

on the boundary $r = a$.
Using the method of separation of variables,

(i) Show that the eigenfunctions have the structure

$$J_n(\lambda_{nm} r) \sin(n\theta + \phi_n) e^{-\lambda_{nm}^2 t}$$

where all terms are to be defined.

(ii) If, at time $t = 0, u(r, \theta, 0)$ is taken to be

$$u(r, \theta, 0) = a - r,$$

show that for subsequent times $u(r, \theta, t)$ can be written

$$u(r, \theta, t) = \sum_{m=0}^{\infty} c_m J_0(\lambda_m r) e^{-\lambda_m^2 t}$$

where c_m and λ_m are to be defined, but are not to be explicitly calculated.

What is the asymptotic behaviour of the solution as $t \to \infty$?
[You may quote without proof the following expression for the Laplacian of u in circular polars:

$$\nabla^2 u \equiv \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}.]$$