Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M2M2

Differential Equations

Date: Tuesday, 9th May 2006 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. The differential operator L is defined by

$$Ly \equiv xy'' - 2y' + (x + \frac{2}{x})\, y\,.$$

(i) Show that $y(x) = x \sin x$ is a solution to the homogenous equation $Ly = 0$.

(ii) Find the Wronskian $W(x)$ for any linearly independent pair of solutions $y_1,\ y_2$ of the equation $Ly = 0$.

(iii) Construct a Green's function appropriate for a boundary value problem where the Green's function vanishes at $x = \frac{\pi}{2}$ and $x = \pi$.

(iv) Use the Green's function to solve the equation

$$Ly = x^2, \quad \frac{\pi}{2} \leq x \leq \pi,$$

with

$$y\left(\frac{\pi}{2}\right) = 1, \quad y(\pi) = 0\,.$$

2. Find two series solutions about $x = 0$ for the equation

$$y'' - xy' + \lambda y = 0\,,$$

where λ is a constant.

What is the radius of convergence for each solution?

Determine the eigenvalues λ for which one or other series terminates as a polynomial; call such solutions the eigenfunction solutions $p_\lambda(x)$.

Rewrite the differential equation in Sturm-Liouville form, and hence deduce the orthogonality relation between the eigenfunction solutions.

3. The function $f(x)$ is defined as follows:

$$f(x) \; = \; e^x, \quad 0 \le x \le \pi,$$

and
$$f(x) = f(-x),$$

and
$$f(x + 2\pi) = f(x)\,.$$

Plot the function $f(x)$ over the range $-2\pi < x < 2\pi$.

Show that the Fourier series for $f(x)$ is

$$f(x) = \frac{e^\pi - 1}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n e^\pi - 1}{1 + n^2} \; \cos nx \; .$$

By considering particular values of $x = 0$ and $x = \pi$, deduce that

$$\tanh\left(\frac{\pi}{2}\right) = \frac{4}{\pi}\left(\frac{1}{2} + \frac{1}{10} + \frac{1}{26} + \dots\right) .$$

4. The Fourier Transform of the function $f(x)$ is defined by

$$\widehat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx \; .$$

State the inverse transform which defines $f(x)$ in terms of $\widehat{f}(k)$.

The function $f(x)$ is defined as

$$f(x) \;\; = \;\; \begin{cases} \frac{1}{2} e^{-x} & , \quad x > 0, \\ \frac{1}{2} e^{x} & , \quad x < 0. \end{cases}$$

Find $\widehat{f}(k)$.

State the convolution theorem and use it to evaluate the integral

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(1 + k^2)^2}\, e^{ikx}\, dk \; .$$

5. A uniform material with a spherical hole of radius a has an axisymmetric temperature distribution governed by Laplace's equation

$$\nabla^2 T = 0, \quad r \geq a,$$

together with the boundary condition that $T(a, \theta) = \sin\theta$ on $r = a$, and $T(r, \theta) \to 0$ as $r \to \infty$.

Using the method of separation of variables as appropriate for spherical polar coordinates, show that the required solution is

$$T(r, \theta) \;=\; \frac{2}{3}\frac{a}{r}\left(1 - \frac{a^2}{r^2}\, P_2(\cos\theta)\right) \;,$$

where $P_2(\cos\theta)$ is a Legendre polynomial.

[You may quote without proof the axisymmetric form of the operator ∇^2 :

$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) \;+\; \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) \;.$$

Legendre's equation for $y(x)$ is

$$(1 - x^2)\,\frac{d^2 y}{dx^2} \;-\; 2x\,\frac{dy}{dx} \;+ l(l+1)\, y = 0,$$

and has the polynomial solutions $P_l(x)$.]