1. The differential operator L is defined by

$$Ly \equiv y'' - 2y' + \left(1 - \frac{2}{x^2}\right) y$$

in $1 < x < \infty$.

(a) Show that $y(x) = x^2 e^x$ is a solution to the homogenous equation $Ly = 0$.

(b) Find a Wronskian for the equation $Ly = 0$ and hence construct a second solution to the homogenous equation.

(c) Using a Green's Function as appropriate for an *initial* value problem, solve

$$Ly = xe^x$$

with $y(1) = y'(1) = 0$.

2. Consider the 2nd order homogenous ODE

$$xy'' + (1-x)y' + ky = 0 \ .$$

(a) Show that it has a regular singular point at $x = 0$.

(b) Find **one** power series solution to the ODE about the point $x = 0$, which is finite at $x = 0$.

(c) What is the radius of convergence of the series solution?

(d) Determine the values of the parameter k, $= \nu$ say, for which the series terminates as a polynomial; call such solutions $L_\nu(x)$.

(e) Rewrite the ODE in Sturm-Liouville form and hence deduce an orthogonality relationship for the different polynomial solutions L_ν and $L_\mu(x)$, $\nu \neq \mu$.

3. Using the complex form of the Fourier series, show that for $0 < x < 2\pi$

$$e^x = \frac{1}{2\pi}\left(e^{2\pi} - 1\right) \sum_{n=-\infty}^{\infty} \frac{e^{inx}}{1 - in} .$$

By considering a particular value of x, deduce that

$$\begin{aligned} \pi \coth \pi &= \pi \frac{(e^{2\pi}+1)}{e^{2\pi}-1} \\ &= 1 + 2\sum_{n=1}^{\infty} \frac{1}{1+n^2} . \end{aligned}$$

Similarly, show that a second choice for x leads to the result

$$\pi \text{ cosech } \pi = 1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} .$$

4. The Fourier Transform of the function $f(x)$ is defined by

$$\widehat{f}(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx .$$

State the inverse transform which defines $f(x)$ in terms of $\widehat{f}(k)$.

The function $f(x)$ is defined as

$$f(x) = \begin{cases} e^{-x} & \text{for } x > 0 , \\ -e^{x} & \text{for } x < 0 . \end{cases}$$

(a) Find $\widehat{f}(k)$.

(b) Use the convolution theorem to find the function $g(x)$ which satisfies

$$\int_{-\infty}^{\infty} f(y) g(x-y) dy = x g(x)$$

and takes the value $g(0) = 1/2$.

5. A uniform sphere has an axisymmetric temperature distribution $T(r,\theta)$ governed by Laplace's equation

$$\nabla^2 T \;=\; 0\,, \qquad 0 \le r \le a, \quad 0 \le \theta \le \pi\,,$$

together with a boundary condition specifying $T(a,\theta)$.

Using the method of separation of variables, show that the solution which is bounded as $r \to 0$ is

$$T(r\theta) \;=\; \sum_{m=0}^{\infty} c_m \left(\frac{r}{a}\right)^m P_m(\cos\theta),$$

where $P_m(\cos\theta)$ are Legendre polynomials.

For the particular case when

$$T(a,\theta) \;=\; \begin{cases} T_0\,, & 0 \le \theta < \pi/2\,, \\ -T_0\,, & \pi/2 < \theta \le \pi, \end{cases}$$

show that the constants c_m are given by

$$c_m \;=\; \begin{cases} 0\,, & m \text{ even}, \\ \dfrac{(2s)!}{2^{2s+1} s!(s+1)!}\,, & m = 2s+1\,. \end{cases}$$

You may quote without proof the axisymmetric form of Laplacian ∇^2 :

$$\nabla^2 \;=\; \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\,.$$

You may also quote the result

$$P'_m(0) \;=\; \begin{cases} 0\,, & m \text{ even.} \\ \dfrac{(m+1)!(-1)^{\frac{m-1}{2}}}{2m\left(\frac{m-1}{2}\right)!\left(\frac{m+1}{2}\right)!}\,, & m \text{ odd.} \end{cases}$$