1. The function $y(x)$ satisfies the inhomogeneous differential equation

$$Ly \equiv xy'' + (2-x)y' - y = e^x \ .$$

(i) Show that $y(x) = 1/x$ is a solution to the homogeneous equation $Ly = 0$.

(ii) Find a Wronskian for the equation $Ly = 0$ and hence construct a second solution to the homogeneous equation.

(iii) Using a Greens function, deduce that the solution satisfying the conditions $y = 0$ and $y' = 0$ at $x = 1$ is

$$y(x) = \left(\frac{x-2}{x}\right) e^x + \frac{e}{x} \ .$$

2. Find the general series solution about $x = 0$ for the equation

$$(1-x^2)y'' - 3xy' + \lambda y = 0$$

where λ is a (real) constant.

(i) What is the radius of convergence for the series?

(ii) Determine the eigenvalues, λ, for which the series terminates as a polynomial. Calling these polynomial eigenfunctions $y_n(x)$, write down the form $y_0(x)$, $y_1(x)$, $y_2(x)$, $y_3(x)$, in each case to within an arbitrary constant.

(iii) Rewrite the equation in Sturm-Liouville form, and hence deduce the orthogonality relation between the eigenfunctions $y_n(x)$.

3. The Fourier Transform of the function $f(x)$ is defined by

$$\widehat{f}(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \; .$$

State the inverse transform which expresses $f(x)$ in terms of $\widehat{f}(k)$.

The function $p(x)$ is defined as

$$p(x) \;=\; \begin{cases} e^{-x} & \text{for} \quad x > 0\,, \\ 0 & \text{for} \quad x < 0\,. \end{cases}$$

(i) Find $\widehat{p}(k)$.

(ii) Use the convolution theorem to find the function $f(x)$ which satisfies the integral equation

$$\int_0^{\infty} p(y)\, f(x-y)\, dy \;=\; x f(x)\,.$$

4. The function $u(x,y)$ satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \;=\; 0$$

in the strip $-\infty < x < \infty$, $0 < y < a$, together with the boundary conditions

$$u(x,0) \;=\; u(x,a) \;=\; \begin{cases} u_0\, e^{-\lambda x}, & x > 0, \\ 0, & x < 0, \end{cases}$$

where λ is a real positive constant.

Show, using a Fourier Transform with respect to x, that the solution for $u(x,y)$ is

$$u(x,y) \;=\; \frac{u_0}{2\pi} \int_{-\infty}^{\infty} \frac{\cosh\{k(y-\frac{a}{2})\}}{\cosh(ka/2)}\, \frac{e^{iky}}{\lambda + ik}\, dk \; .$$

What does this reduce to in the region $0 < y \ll a$, $a \to \infty$?

You are not required to evaluate the resulting integral.

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5. The function $u(r,\theta,t)$ satisfies the diffusion equation

$$\nabla^2 u = \frac{\partial u}{\partial t}$$

within the circle $r < a$, for time $t > 0$. It is nonsingular at the origin, and takes the value

$$u(a,\theta,t) = 0$$

on the boundary $r = a$.

(i) Show that the eigenmode solutions have the structure

$$J_n(\lambda_{mn} r)\,\sin(n\theta + \phi_n)\,e^{-\lambda_{mn}^2 t}\ ,$$

where all terms are to be defined.

(ii) If, at time $t = 0$, $u(r,\theta,0)$ is taken to be

$$u(r,\theta,0) = a - r\ ,$$

show that for subsequent times $u(r,\theta,t)$ can be written

$$u(r,\theta,t) = \sum_{m=0}^{\infty} c_m\,J_0(\lambda_m r)\,e^{-\lambda_m^2 t}\ ,$$

where c_m are to be defined, but integral expressions are not to be explicitly evaluated.

You may quote, without proof, the following expression for the Laplacian of u in circular polars

$$\nabla^2 u \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\ .$$