Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2007

This paper is also taken for the relevant examination for the Associateship.

M2M1

Vector field theory

Date: Monday, 9th May 2007 Time: 10 am – 12 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (a) Using tensor calculus simplify

$$\mathbf{a} \cdot (\mathbf{a} \times \mathbf{c})$$

where $\mathbf{a}, \mathbf{c}$ are constant vectors (you may assume that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$).

(b) Using tensor calculus, for constant vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, show that

$$(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a}) = \mathbf{c}(\mathbf{a} \cdot [\mathbf{b} \times \mathbf{c}])$$

and hence that

$$(\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})] = (\mathbf{a} \cdot [\mathbf{b} \times \mathbf{c}])^2.$$

(c) Evaluate $\nabla^2[\nabla \cdot (\mathbf{r}/r^2)]$ where $\mathbf{r} = (x, y)$ and $r = |\mathbf{r}|$.

2. (a) State without proof Green's theorem in the plane where C is a simple closed curve in the plane enclosing an area (surface) A.

(b) Given a closed simple curve C in the x, y plane enclosing an area A take each of the following and demonstrate whether it is true or show that it is false.

(i) The area is

$$\oint_C x dy. \quad \text{(true or false? Show your reasoning.)}$$

(ii) The area is

$$\oint_C -y dx. \quad \text{(true or false? Show your reasoning.)}$$

(iii) The area is

$$\frac{1}{2} \oint_C (x dy - y dx). \quad \text{(true or false? Show your reasoning.)}$$

(c) Calculate the area of the hypocycloid defined by $x^{2/3} + y^{2/3} = a^{2/3}$. Hint use a parametrisation of the curve: $x(t) = a\cos^3 t, y = a\sin^3 t$ with $0 \le t \le 2\pi$.

(d) Consider the following path C: C is the union of a circle of radius 4 center the origin traversed anti-clockwise and another concentric circle of radius 1 traversed clockwise. Evaluate the following path integral directly

$$\int_C (3x - y)dx + x dy$$

and then verify your result using Green's theorem. Be careful to justify your use of Green's theorem.

3. (a) The region R is defined as the finite area enclosed by the lines $y = x/2$, $y = 0$ and the curve $x = 1 + y^2$. Sketch the region R.

(b) Using the change of variables $x = u^2 + v^2$, $y = uv$ show that the transformed region R in the u, v domain is a triangle: pay particular attention to the mapped positions of the vertices.

(c) Calculate the area of the region R.

(d) Is the coordinate system defined by u, v orthogonal? (Justify your answer.)

4. Using the divergence theorem deduce Green's first identity for two scalar fields ϕ and ψ:

$$\int_V [\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi]dV = \int_S \phi(\nabla\psi\cdot\mathbf{n})dS$$

(you should define V, S and $\mathbf{n}$). Thence deduce Green's second identity:

$$\int_V [\phi\nabla^2\psi - \psi\nabla^2\phi]dV = \int_S (\phi\psi_n - \psi\phi_n)dS.$$

The scalar fields ϕ and ψ satisfy the Helmholtz equations

$$\nabla^2\psi + k^2\psi = \delta(\mathbf{x} - \mathbf{x}'), \quad \nabla^2\phi + k^2\phi = f(\mathbf{x})$$

where k is constant. If the boundary conditions on S are that $\psi = 0$ and $\phi = q(\mathbf{x})$ on S then show that at any point within the surface S ϕ is given by:

$$\phi(\mathbf{x}) = \int_V \psi f(\mathbf{x})dV + \int_S q(\mathbf{x})\psi_n dS.$$

If the boundary conditions change so that $\phi_n = p(\mathbf{x})$ on a portion of the surface S_a and $\phi = q(\mathbf{x})$ on the remaining portion of the surface S_b find $\phi(\mathbf{x})$.

5. Consider the following Poisson equation in spherical polar coordinates (r, θ, λ) with axisymmetry assumed, i.e no dependence on the azimuthal angle λ,

$$\nabla^2 \phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) = r^2 \cos^2\theta. \qquad (*)$$

Consider a solution written in the form of a complementary function plus a particular integral,

$$\phi = \phi_{\mathrm{CF}} + \phi_{\mathrm{PI}},$$

and suppose that this solution remains bounded in the domain $r \leq a$.

(a) Write down ϕ_{CF}.

(b) Find a particular integral ϕ_{PI}.

(c) Hence find the solution to $(*)$ in the domain $r \leq a$ that satisfies the boundary condition $\phi = 0$ on $r = a$.

[You may use the following formulae for the first few Legendre polynomials: $P_0(c) = 1$, $P_1(c) = c$, $P_2(c) = \frac{1}{2}(3c^2 - 1)$.]