Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M2M1

Vector Field Theory

Date: Monday, 8th May 2006 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (a) Using tensor calculus prove the identities

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A},$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}),$$

for vector fields $\mathbf{A}$ and $\mathbf{B}$.

(b) Two vector fields $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{H}(\mathbf{x}, t)$ obey the following equations:

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \cdot \mathbf{H} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t}, \quad \nabla \times \mathbf{H} = \frac{\partial \mathbf{E}}{\partial t}.$$

Show that $\mathbf{E}$ and $\mathbf{H}$ satisfy

$$\nabla^2 \mathbf{E} = \frac{\partial^2 \mathbf{E}}{\partial t^2}, \quad \nabla^2 \mathbf{H} = \frac{\partial^2 \mathbf{H}}{\partial t^2}$$

and that

$$\nabla \cdot (\mathbf{H} \times \mathbf{E}) = \frac{1}{2} \frac{\partial}{\partial t} (|\mathbf{E}|^2 + |\mathbf{H}|^2).$$

2. (a) Find an equation for the tangent plane to the surface $2(x^2 + y^2) + z^2 y = 14$ at the point $(2, 1, 2)$.

(b) Find the cosine of the angle between the surfaces $2(x^2 + y^2) + z^2 y = 14$ and $z = 2(x^2 + y^2) - 8$ at the point $(2, 1, 2)$.

(c) Let $\widehat{\mathbf{p}}$ be a unit vector and

$$\frac{\partial \phi}{\partial p} = \widehat{\mathbf{p}} \cdot \nabla \phi$$

be the directional derivative of ϕ with respect to $\widehat{\mathbf{p}}$. In what direction from the point $(2, 1, 2)$ is the directional derivative of $\phi = 2(x^2 + y^2) + z^2 y - 14$ a maximum? What is the magnitude of this maximum?

(d) Let $\mathbf{A}$ be the vector field

$$\mathbf{A} = (4x, 4y + z^2, 2zy).$$

Evaluate the line integral

$$\int_C \mathbf{A} \cdot d\mathbf{r}$$

where C is the path connecting $(0, 0, 0)$ to $(1, 1, 1)$ with the path following the parametric curve $x = t, y = t^2, z = t^3$ with $0 \leq t \leq 1$.

3. (a) State without proof Stokes' theorem for a vector field $\mathbf{A}$ with continuous derivatives, an open surface S bound by a closed (non-intersecting) curve C.

Hence, or otherwise, evaluate

$$\int\int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n} dS$$

where $\mathbf{A} = (4x^2 + y - 3)\mathbf{i} + 5xy\mathbf{j} + (xz + z^2)\mathbf{k}$ and S is
(i) the surface of the hemisphere $x^2 + y^2 + z^2 = 9$ in $z \geq 0$ and
(ii) the paraboloid $z = 1 - (x^2 + y^2)$ in $z \geq 0$.

(b) State without proof the divergence theorem satisfied by a differentiable function $\mathbf{A}$ in a simply connected volume V bounded by a surface S.

Hence or otherwise evaluate

$$\int\int_S \mathbf{A} \cdot \mathbf{n} dS$$

where $\mathbf{A} = (4x, 4y + z^2, 2zy)$ and S is the surface of the unit cube $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$.

4. Curvilinear coordinates are defined by u, v, z in terms of Cartesian coordinates x, y, z by

$$x = a \cosh u \cos v, \quad y = a \sinh u \sin v, \quad z = z$$

where a is a constant, $u \geq 0$ and $0 \leq v \leq 2\pi$.

(a) Sketch the loci in the x, y plane to show the curves of $u =$ constant and $v =$ constant. Under what circumstances would this be a useful set of coordinates?

(b) Show that the scale factors h_1, h_2 and h_3 for the coordinates u, v, z are given by

$$h_1 = h_2 = a\sqrt{\sinh^2 u + \sin^2 v}, \quad h_3 = 1.$$

(c) Using the general result

$$\nabla^2 \phi = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial q_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \phi}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial \phi}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \phi}{\partial q_3} \right) \right)$$

for scale factors h_i and orthogonal coordinates q_i $(i = 1, 2, 3)$, determine Laplace's equation for ϕ in terms of u, v, z.

(d) If $\phi(u, v) = \phi(u^2 + v^2)$, i.e. ϕ is a function only of $(u^2 + v^2)$, show that the general solution of Laplace's equation is

$$\phi(u, v) = A \log(u^2 + v^2) + B$$

with A and B being arbitrary constants.

5. In spherical polar coordinates (r, θ, λ), axially symmetric functions do not depend upon λ. State without proof, the set of separable solutions, that are axially symmetric, of the Laplace equation $\nabla^2 \phi = 0$ where $\phi = \phi(r, \theta)$.

(a) Given the generating function

$$[1 - 2ch + h^2]^{\frac{-1}{2}} = \sum_{n=0}^{\infty} P_n(c) h^n, \qquad |h| < 1,$$

find $P_0(c), P_1(c)$ and $P_2(c)$.

(b) The temperature field T within $a \leq r \leq b$ satisfies $\nabla^2 T = 0$. The spherical surfaces at $r = a, b$ are subjected to the following boundary conditions:

$$T = 1 \quad \text{on} \quad r = b$$

and

$$T = T_0 \cos\theta \quad \text{on} \quad r = a.$$

Show that the temperature field in $a \leq r \leq b$ is given by

$$T = \frac{b}{(b-a)} \left(1 - \frac{a}{r}\right) + \frac{T_0 a^2}{(a^3 - b^3)} \left(r - \frac{b^3}{r^2}\right) \cos\theta.$$

(c) Prove that the solution you have just obtained is the unique (i.e the one and only) solution to the stated boundary value problem.