Imperial College London

UNIVERSITY OF LONDON

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BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2005

This paper is also taken for the relevant examination for the Associateship.

M2M1 Vector Field Theory

Date: examdate Time: examtime

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

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1. (a) Show that the vector field

$$\mathbf{A} = xy^2 \exp(x^2)\mathbf{i} + y \exp(x^2)\mathbf{j} + z^2\mathbf{k}$$

is the gradient of a scalar function $\phi(x, y, z)$. Determine ϕ. Hence or otherwise evaluate the line integral

$$\int_C \mathbf{A} \bullet d\mathbf{r}$$

where C is any curve from the origin to the point $(1, 1, 1)$.

(b) Evaluate the integral

$$\int_{C_1} \mathbf{B} \bullet d\mathbf{r}$$

where C_1 is a straight line from the origin to the point (1,1,1) and

$$\mathbf{B} = \mathbf{A} + z\mathbf{i} + x^2\mathbf{j} + y^3\mathbf{k} \quad .$$

2. (a) State the divergence theorem. Then by substituting $\boldsymbol{F}=\boldsymbol{u} \times \boldsymbol{K}$ into the theorem, where $\boldsymbol{K}$ is an arbitrary constant vector, prove that

$$\int_V \nabla \times \boldsymbol{u} \, dV = \int_S \boldsymbol{n} \times \boldsymbol{u} \, dS \ ,$$

where V is a volume bounded by a surface S with outward normal $\boldsymbol{n}$.

(b) State Stokes' Theorem. Then use the theorem to show that

$$\int\int_S (\nabla \times \mathbf{F}) \bullet \mathbf{n} dS = 2\pi$$

where

$$\mathbf{F} = (x - 2y)\mathbf{i} + (y - \frac{1}{2}yz^2)\mathbf{j} + (z - \frac{1}{2}y^2 z)\mathbf{k}$$

and $\mathbf{n}$ is the unit outward normal vector to the hemisphere S given by $x^2 + y^2 + z^2 = 1$ and $z \geq 0$.

3. **(a)** Curvilinear coordinates (u, v, w) are defined in terms of the cartesian coordinates (x_1, x_2, x_3) by the relations

$$x_1 = e^u \cos v \quad x_2 = e^u \sin v \quad x_3 = w$$

with $-\pi < w \leq \pi$. Using the identities

$$\begin{aligned} \delta\mathbf{x} &= (h_1 \delta u)\mathbf{e_1} + (h_2 \delta v)\mathbf{e_2} + (h_3 \delta w)\mathbf{e_3} \\ \delta\mathbf{x} &= \delta x_1 \mathbf{i} + \delta x_2 \mathbf{j} + \delta x_3 \mathbf{k} \end{aligned}$$

or otherwise, calculate the scale factors h_1, h_2, and h_3. Express each of the unit vectors $\mathbf{e_1}$, $\mathbf{e_2}$ and $\mathbf{e_3}$ in terms of the cartesian unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

(b) A scalar field ϕ is given in terms of the curvilinear coordinates by

$$\phi = e^{2u} - 3w^2 \quad .$$

Find $\mathbf{F} = \nabla\phi$ and the scalar field $\nabla^2\phi$ in terms of the curvilinear coordinates. Then express $\mathbf{F}$ in terms of the cartesian coordinates and re-calculate $\nabla^2\phi$ in terms of the cartesian coordinate system. Show that the two expressions for $\nabla^2\phi$ are consistent.

You may quote, without proof, the following results for a generalized coordinate system (q_1, q_2, q_3)

$$\nabla\phi = \frac{1}{h_1}\frac{\partial\phi}{\partial q_1}\mathbf{e_1} + \frac{1}{h_2}\frac{\partial\phi}{\partial q_2}\mathbf{e_2} + \frac{1}{h_3}\frac{\partial\phi}{\partial q_3}\mathbf{e_3} \quad .$$

$$\nabla\cdot\mathbf{F}(q_1, q_2, q_3) = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(h_2 h_3 F_1) + \frac{\partial}{\partial q_2}(h_3 h_1 F_2) + \frac{\partial}{\partial q_3}(h_1 h_2 F_3)\right] \quad .$$

4. (a) In spherical polar coordinates (r, θ, λ), axially symmetric functions do not depend on λ. The potential ϕ is subject to the Poisson equation

$$\nabla^2 \phi = 24 r \cos^2 \theta \,, \quad \text{for} \quad r \le a \,,$$

together with the boundary condition

$$\phi = 0 \,, \quad \text{at} \quad r = a \quad .$$

Given that

$$\cos^2 \theta = \frac{1}{3} P_0(c) + \frac{2}{3} P_2(c)$$

where $c = \cos \theta$ and $P_0(c)$, $P_2(c)$ are Legendre polynomials, find a particular solution of the Poisson equation in the form $\phi_P = r^3 (K_0 P_0(c) + K_2 P_2(c))$. Then solve for $\phi(r, \theta)$ in given region $r \le a$.

You may quote, without proof, the results: $P_0(c) = 1$, $P_1(c) = c$ and $P_2(c) = \frac{1}{2}(3c^2 - 1)$. Also, it may be assumed that the Laplacian of a radially symmetric function ϕ in terms of r and c is

$$\nabla^2 \phi(r) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial c} \left((1 - c^2) \frac{\partial \phi}{\partial c} \right) \quad .$$

(b) Let (ρ, θ, z) be the usual cylindrical coordinates and consider the Laplace equation $\nabla^2 \phi = 0$. Assume that the solution ϕ has the form $\phi(\rho, \theta, z) = F(\rho) G(\theta)$ (note that ϕ is $z-$independent).

Deduce the ordinary differential equations that F and G must satisfy (do not solve them!).

You may quote, without proof, that the Laplacian in cylindrical coordinates of a function ϕ is

$$\nabla^2 \phi = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} \quad .$$

5. **(a)** Prove the identity

$$T'_{ij}T'_{ij} = T_{ij}T_{ij} \quad ,$$

where T'_{ij} and T_{ij} denote the components of a cartesian tensor with respect to right handed cartesian coordinate systems S' and S.

(b) State the transformation law satisfied by a T_{ijk}, a cartesian tensor of rank 3. Show that T_{iij} is a cartesian tensor of rank 1.

(c) Give the definition of the inner product of two cartesian tensors. Then verify that the inner product of two cartesian tensors of order 2 is a cartesian tensor of order 2.