1. (a) The vector field

$$\mathbf{F} = 3(x^2 + z^2)\mathbf{i} + (2z^2 y)\mathbf{j} + (6xz + 2y^2 z)\mathbf{k}$$

is the gradient of a scalar function $\phi(x, y, z)$. Determine ϕ. Hence or otherwise evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where C is a curve from the origin to the point $(1, 1, 1)$.

(b) Calculate the directional derivative of ϕ at the point (1,1,1) in the direction (2,2,-2).

(c) Evaluate the integral

$$\int_{C_1} \mathbf{A} \cdot d\mathbf{r}$$

where C_1 is a straight line from the origin to the point (1,1,1) and

$$\mathbf{A} = \mathbf{F} + 4y^3\mathbf{i} \quad .$$

2. (a) State, without proof, the divergence theorem satisfied by a differentiable vector field $\mathbf{E}$ in a simply connected volume V bounded by a surface S. For the vector field

$$\mathbf{G} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad ,$$

what is $\nabla \cdot \mathbf{G}$? Verify the divergence theorem for the vector field $\mathbf{G}$ when the simply connected volume is a sphere of radius a centered at the origin.

(b) Using Stokes' Theorem show that

$$\iint_S (\nabla \times \mathbf{H}) \cdot \mathbf{n} dS = 2\pi$$

where

$$\mathbf{H} = -y\mathbf{i} + x\mathbf{j} + \sin(z)\mathbf{k}$$

and $\mathbf{n}$ is the unit outward normal vector to the curved surface S of the hemisphere $x^2 + y^2 + z^2 \leq 1$ with $z \geq 0$.

3. (a) Curvilinear coordinates (u, v, w) are defined in terms of the cartesian coordinates (x_1, x_2, x_3) by the relations

$$x_1 = uv\cos w \quad x_2 = uv\sin w \quad x_3 = \frac{1}{2}(u^2 - v^2)$$

with $-\pi < w \leq \pi$. Using the identities

$$\begin{aligned} \delta\mathrm{x} &= (h_1\delta u)\mathbf{e_1} + (h_2\delta v)\mathbf{e_2} + (h_3\delta w)\mathbf{e_3} \\ \delta\mathrm{x} &= \delta x_1\mathbf{i} + \delta x_2\mathbf{j} + \delta x_3\mathbf{k} \end{aligned}$$

or otherwise, calculate the scale factors h_1, h_2, and h_3. Express each of the unit vectors $\mathbf{e_1}$, $\mathbf{e_2}$ and $\mathbf{e_3}$ in terms of the cartesian unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

(b) A scalar field ϕ is given in terms of the curvilinear coordinates by

$$\phi = \frac{1}{4}(u^2 + v^2)^2 \quad .$$

Find $\nabla\phi$ in terms of $u, v, w, \mathbf{e_1}, \mathbf{e_2}$ and $\mathbf{e_3}$. Express the function ϕ in terms of the cartesian coordinates and re-calculate $\nabla\phi$ in terms of the cartesian coordinate system. Show that the two expressions for $\nabla\phi$ are consistent.

You may quote, without proof, the following result for a generalized coordinate system (q_1, q_2, q_3)

$$\nabla f(q_1, q_2, q_3) = \frac{1}{h_1}\frac{\partial f}{\partial q_1}\mathbf{e_1} + \frac{1}{h_2}\frac{\partial f}{\partial q_2}\mathbf{e_2} + \frac{1}{h_3}\frac{\partial f}{\partial q_3}\mathbf{e_3} \quad .$$

4. In spherical polar coordinates (r, θ, λ), axially symmetric functions do not depend on λ. State, without proof, the set of separable solutions, which are axially symmetric, of the Laplace equation $\nabla^2\psi = 0$, where $\psi = \psi(r, \theta)$.

The function $\phi(r, \theta)$ satisfies the Poisson equation

$$\nabla^2\phi = 3\,, \quad \text{for} \quad r \leq a\,,$$

together with the boundary condition

$$\phi = a^2\cos(2\theta)\,, \quad \text{at} \quad r = a \quad .$$

Show that a particular solution in the region $r \leq a$ of the Poisson equation is $\phi_p = Kr^2$ where the constant K should be determined. Show that $a^2\cos(2\theta)$ can be expressed as a linear combination of the Legendre polynomials P_0, $P_1(c)$, and $P_2(c)$ with $c = \cos\theta$.

Determine the solution for $\phi(r, \theta)$ in the region $r \leq a$.

You may quote, without proof the results: $P_0 = 1$, $P_1(c) = c$ and $P_2(c) = \frac{1}{2}(3c^2 - 1)$. Also, it may be assumed that the Laplacian of a radially symmetric function $\phi_p(r)$ is

$$\nabla^2\phi_p(r) = \frac{d^2\phi_p(r)}{dr^2} + \frac{2}{r}\frac{d\phi_p(r)}{dr} \quad .$$

5. (a) State the transformation law satisfied by a tensor of rank 2. Prove the identity

$$T'_{ij}T'_{ij} = T_{ij}T_{ij} \quad ,$$

where T'_{ij}, T_{ij} denote the components of a tensor with respect to right handed cartesian coordinate systems S' and S.

(b) Use tensor notation to establish the identities

(i) $\nabla \cdot (\phi\mathbf{u}) = (\nabla\phi) \cdot \mathbf{u} + \phi(\nabla \cdot \mathbf{u})$

(ii) $\nabla \times (\phi\mathbf{u}) = (\nabla\phi) \times \mathbf{u} + \phi(\nabla \times \mathbf{u})$

(iii) $\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{v})$