1. The partial differential equation governing the fluid speed $u(x,t)$ in a one-dimensional flow is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0.$$

Initially, at $t = 0$, the flow is given by $u(x, 0) = e^{-x^2}$.

(i) Show that an implicit form of the solution for $u(x,t)$ is

$$x = ut \pm [\log(1/u)]^{1/2}.$$

(ii) At some time t_s, a shock forms in the solution to this problem at a point $x = x_s$. Find t_s and x_s and the corresponding value of u at this point.

2. (i) The Lagrangian description of a one-dimensional flow is given implicitly by the equation

$$e^{-x} = e^{-t} + e^{-\zeta} - 1$$

where ζ labels the fluid particle which is at $x = \zeta$ at $t = 0$. Find the speed, at time $t = 1$, of the fluid particle which is at $x = 1$ when $t = 0$.

(ii) Solve the following partial differential equation by the method of characteristics:

$$x\frac{\partial u}{\partial t} + t\frac{\partial u}{\partial x} = 0, \quad \text{with } u = e^{-x^2} \text{ when } t = 0.$$

(iii) The kinematic wave equation for $u(x,t)$ is

$$\frac{\partial u}{\partial t} + (u+k)\frac{\partial u}{\partial x} = 0$$

where k is a positive constant. Suppose initial conditions for this partial differential equation are

$$u(x,0) = \left\{ \begin{array}{ll} 1, & x \leq 0, \\ 1+x, & 0 \leq x \leq 1, \\ 2, & x \geq 1. \end{array} \right\}$$

Sketch the characteristics for this equation in the (x,t)-plane.

3. (i) Derive the Euler-Lagrange equation for the function $u(x, y)$ which is the extremum of the functional

$$\int_\Omega \left(\frac{1}{2}\left(u_x^2 + u_y^2\right) - f(u)\right) dxdy.$$

The domain Ω is assumed to be a simply connected finite subset of $\mathbb{R}^2$ with smooth boundary.

(ii) Consider the Hamiltonian

$$H(q, p) = \frac{1}{2}e^{-bt}p^2 + \frac{1}{2}e^{bt}\omega^2 q^2.$$

(a) Derive Hamilton's equations of motion.

(b) Show that the equations of motion are equivalent to the equation

$$\ddot{q} + b\dot{q} + \omega^2 q = 0.$$

(iii) Consider a system of two particles moving in one dimension, with Lagrangian

$$L = \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2\dot{q}_2^2 - V(q_1, q_2),$$

where the interaction potential depends only on the difference $q_1 - q_2$:

$$V(q_1, q_2) = V(q_1 - q_2).$$

(a) Derive the Euler-Lagrange equations.

(b) Show that the total momentum

$$P = m_1\dot{q}_1 + m_2\dot{q}_2$$

is conserved.

4. Consider the Lagrangian

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + A(x,y)\dot{x} + B(x,y)\dot{y} - \Phi(x,y).$$

(i) Derive the Euler-Lagrange equations.

(ii) Show that the Euler-Lagrange equations remain invariant if we replace $A(x,y),\ B(x,y)$ with

$$\hat{A} = A + \frac{\partial f}{\partial x}, \quad \hat{B} = B + \frac{\partial f}{\partial y},$$

where $f = f(x,y)$ is an arbitrary function.

(iii) Derive the Hamiltonian function.

(iv) Let $\Phi(x,y) = \sin(x) + \sin(y),\ A(x,y) = \cos(x)\sin(y),\ B(x,y) = \sin(x)\cos(y)$. Show that in this case the Hamiltonian system is integrable.