Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2008

This paper is also taken for the relevant examination for the Associateship.

M2AA3 ORTHOGONALITY

Date: ? 2008 Time: ?

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Statistical tables will not be available.

1. (a) Let

$$A = \frac{1}{4}\begin{pmatrix} 1 & 3 & 1 \\ 3 & 25 & 3 \\ 1 & 3 & 5 \end{pmatrix},$$

and

$$\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \underline{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \underline{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Assuming that A is positive definite, let $\langle \cdot, \cdot \rangle_A$ be the inner product on $\mathbb{R}^3 \times \mathbb{R}^3$ induced by A. Apply the classical Gram-Schmidt algorithm using $\langle \cdot, \cdot \rangle_A$ to the vectors $\{\underline{e}_i\}_{i=1}^3$ to produce orthonormal vectors $\{\underline{q}_i\}_{i=1}^3$ with respect to $\langle \cdot, \cdot \rangle_A$.

(b) Let $\underline{u} \in \mathbb{R}^n$ with $\|\underline{u}\| = 1$. Define

$$P = I - 2\,\underline{u}\,\underline{u}^T \in \mathbb{R}^{n \times n}.$$

Prove that P is a symmetric orthogonal matrix such that

$$\text{(i)} \quad P\,\underline{u} = -\underline{u}\,,$$

$$\text{and} \quad \text{(ii)} \quad P\,\underline{v} = \underline{v} \qquad \forall\, \underline{v} \in \mathbb{R}^n \quad \text{with} \quad \underline{v}^T\underline{u} = 0\,.$$

Find $\displaystyle\sum_{i=1}^{n} P_{ii}\,.$

(c) Find the Newton form of the cubic interpolating polynomial for the data $\{(x_i, f_i)\}_{i=0}^3 \equiv \{(1,1),\, (-1,5),\, (2,11),\, (0,1)\,\}\,.$

2. (a) Use Givens rotations to compute the least squares solution $\underline{x}^\star$ of the overdetermined linear system $A\,\underline{x} = \underline{b}$, where

$$A = \begin{pmatrix} 4 & 1 \\ 0 & 1 \\ 3 & 2 \end{pmatrix}, \qquad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \text{and} \qquad \underline{b} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}.$$

Calculate the error $\|A\,\underline{x}^\star - \underline{b}\|$, where $\|\underline{y}\| = (\underline{y}^T\,\underline{y})^{\frac{1}{2}}$.

Check your solution by solving the corresponding normal equations.

(b) Define the Cholesky factorization of a symmetric positive definite matrix.

Show that

$$\begin{pmatrix} 16 & 4 & -4 \\ 4 & 10 & 2 \\ -4 & 2 & 27 \end{pmatrix}$$

is positive definite.

Compute its Cholesky factorization.

3. Let V be a real vector space. State the properties that a real-valued function $\langle \cdot, \cdot \rangle$ on $V \times V$ must satisfy for it to be an inner product.

Let U be a subspace of V with basis $\{\phi_i\}_{i=1}^n$. Given any $v \in V$, let

$$E(\underline{\lambda}) = \|v - \sum_{i=1}^{n} \lambda_i \, \phi_i\|^2, \qquad \text{where} \qquad \underline{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_n)^T \in \mathbb{R}^n$$

and $\| \cdot \| = [\langle \cdot, \cdot \rangle]^{\frac{1}{2}}$. Show that

$$E(\underline{\lambda}) = \|v\|^2 - 2\,\underline{\lambda}^T \underline{\mu} + \underline{\lambda}^T G\, \underline{\lambda},$$

where $\underline{\mu} \in \mathbb{R}^n$ and $G \in \mathbb{R}^{n \times n}$ are such that

$$\mu_i = \langle v, \phi_i \rangle \qquad \text{and} \qquad G_{ij} = \langle \phi_i, \phi_j \rangle, \qquad i,\, j = 1 \to n.$$

Derive the following results.

(i) G is a symmetric positive definite matrix.

(ii) There exists a unique $\underline{\lambda}^\star \in \mathbb{R}^n$ such that $G\,\underline{\lambda}^\star = \underline{\mu}$.

(iii) $E(\underline{\lambda}^\star + \underline{h}) = E(\underline{\lambda}^\star) + \underline{h}^T G\, \underline{h} \geq E(\underline{\lambda}^\star) \quad \forall\ \underline{h} \in \mathbb{R}^n$.

(iv) $u^\star = \sum_{i=1}^{n} \lambda_i^\star \, \phi_i \in U$ is such that $\langle v - u^\star, u \rangle = 0 \quad \forall\ u \in U$.

(v) $E(\underline{\lambda}^\star) = \|v\|^2 - \|u^\star\|^2$.

Let $V = C[0,1]$ with inner product

$$\langle f, g \rangle = \int_0^1 (1+x)\, f(x)\, g(x)\, dx \quad \forall\ f,\, g \in C[0,1].$$

Let $U = \mathbb{P}_1$, with basis $\{1, x\}$.

If $v = \frac{13}{6}\,(1+x)^{-1}$, find the corresponding $u^\star$ and calculate $E(\underline{\lambda}^\star)$.

4. (a) For $n \geq 0$, let

$$V_{n+1}(x) = \sin(\cos^{-1} x)\ \sin(n(\cos^{-1} x)) \qquad \forall\ x \in [-1,1].$$

By introducing the change of variable $x = \cos\theta$, and noting a trigonometric identity, derive the recurrence relation

$$V_{n+1}(x) + V_{n-1}(x) = 2\,x\,V_n(x) \qquad \text{for } n \geq 2.$$

Hence, or otherwise, show that $V_{n+1} \in \mathbb{P}_{n+1}$ for $n \geq 1$.

For $n \geq 1$, let

$$x_j^\star = \cos(\frac{j\,\pi}{n}), \qquad j = 0 \to n.$$

Show for $n \geq 1$ that

$$V_{n+1}(x_j^\star) = 0 \qquad \text{and} \qquad V'_{n+1}(x_j^\star) = (-1)^{j+1}\,n\,a_j \qquad j = 0 \to n,$$

where $a_0 = a_n = 2$ and $a_j = 1$ for $j = 1 \to n-1$.

(b) For $i = 0 \to n$, let

$$\ell_i(x) = \prod_{j=0,\,j\neq i}^{n} \frac{(x - x_j)}{(x_i - x_j)},$$

where $\{x_j\}_{j=0}^n$ are distinct points.

Write down the polynomial $p_n \in \mathbb{P}_n$, in terms of $\{\ell_i\}_{i=0}^n$, which interpolates the data $\{x_j,\, f(x_j)\}_{j=0}^n$.

Show that this interpolating polynomial is unique, and hence that $\sum_{i=0}^{n} \ell_i(x) = 1$.

For $i = 0 \to n$, show that

$$\ell_i(x) = \frac{q_i\,L(x)}{(x - x_i)}, \qquad \text{where} \qquad L(x) = \prod_{j=0}^{n}(x - x_j) \quad \text{and} \quad q_i = [L'(x_i)]^{-1}.$$

Hence deduce that

$$p_n(x) = \frac{\sum_{i=0}^{n} q_i\,(x - x_i)^{-1}\,f(x_i)}{\sum_{i=0}^{n} q_i\,(x - x_i)^{-1}} \qquad \text{for} \quad x \neq x_j, \quad j = 0 \to n.$$

Finally, if the interpolation points $\{x_j\}_{j=0}^n$ are chosen so that $x_j = x_j^*$, $j = 0 \to n$, as defined in part (a) above; show that

$$p_n(x) = \frac{\sum_{i=0}^{n} c_i\,(x - x_i^\star)^{-1}\,f(x_i^\star)}{\sum_{i=0}^{n} c_i\,(x - x_i^\star)^{-1}} \qquad \text{for} \quad x \neq x_j^\star, \quad j = 0 \to n,$$

where $c_0 = \frac{1}{2}$, $c_n = (-1)^n\,\frac{1}{2}$ and $c_j = (-1)^j$ for $j = 1 \to n-1$.