M2AA2 – MULTIVARIABLE CALCULUS

SOLUTIONS – PROBLEM SHEET 7

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1 (a)

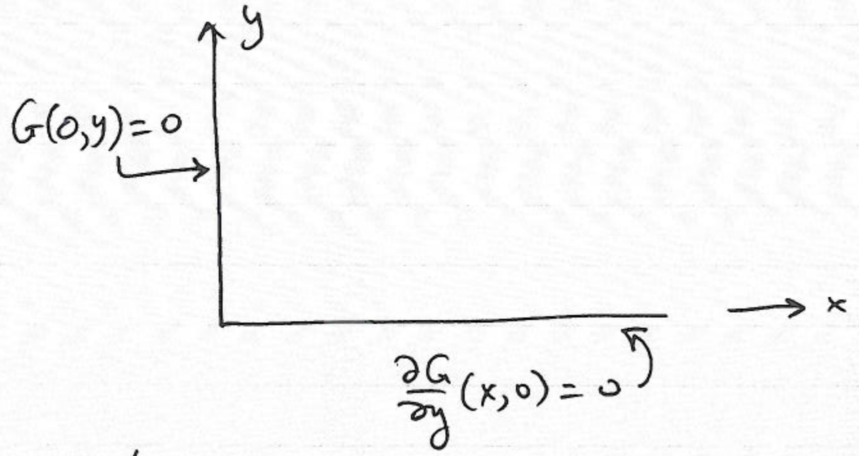

Image system

Source at $\underline{x}_0 = (x_0, y_0)$

Source at $\underline{x}_0' = (x_0, -y_0)$

Sink at $\underline{x}_0'' = (-x_0, -y_0)$

Sink at $\underline{x}_0''' = (-x_0, y_0)$

$$G(\underline{x};\underline{x}_0) = \frac{1}{2\pi}\log\left\{\frac{|\underline{x}-\underline{x}_0|\,|\underline{x}-\underline{x}_0'|}{|\underline{x}-\underline{x}_0''|\,|\underline{x}-\underline{x}_0'''|}\right\}$$

(b) The solution once G is found is given by

$$\phi(\underline{x}_0) = \iint G(\underline{x};\underline{x}_0) f(\underline{x})\, d\underline{x} + \int_C \left(\phi\frac{\partial G}{\partial n} - G\frac{\partial \phi}{\partial n}\right) ds$$

Specialise to this case to find

$$\phi(\underline{x}_0) = \int_0^\infty\!\!\int_0^\infty G(\underline{x};\underline{x}_0) f(x,y)\,dx\,dy + \int_0^\infty q(y)\left[-\frac{\partial G}{\partial x}\right]_{x=0}(0,y;\underline{x}_0)\,dy$$

$$+ \int_0^\infty G(x,0;\underline{x}_0)\left(-\frac{\partial\phi}{\partial y}\right)_{y=0} dx$$

$$= \int_0^\infty\!\!\int_0^\infty G f\,dx\,dy - \int_0^\infty q(y)\frac{\partial G}{\partial x}(0,y;\underline{x}_0)\,dy + \int_0^\infty G(x,0;\underline{x}_0)\,p(x)\,dx$$

where you need $\frac{\partial G}{\partial x}$, G etc. To do this write G in full,

$$G(x,y;x_0,y_0) = \frac{1}{4\pi} \log\left[\frac{[(x-x_0)^2+(y-y_0)^2][(x-x_0)^2+(y+y_0)^2]}{[(x+x_0)^2+(y+y_0)^2][(x+x_0)^2+(y-y_0)^2]} \right]$$

$$\Rightarrow G(x,0;x_0,y_0) = \frac{1}{8\pi} \log\left[\frac{(x-x_0)^2+y_0^2}{(x+x_0)^2+y_0^2} \right]$$

$$\frac{\partial G}{\partial x} = \frac{1}{4\pi}\left(\frac{2(x-x_0)}{(x-x_0)^2+(y-y_0)^2} + \frac{2(x-x_0)}{(x-x_0)^2+(y+y_0)^2} - \frac{2(x+x_0)}{(x+x_0)^2+(y+y_0)^2} - \frac{2(x+x_0)}{(x+x_0)^2+(y-y_0)^2} \right)$$

$$\frac{\partial G}{\partial x}(0,y;x_0) = -\frac{1}{\cancel{4}\pi}\left[\frac{x_0}{x_0^2+(y-y_0)^2} + \frac{x_0}{x_0^2+(y+y_0)^2} \right]$$

These go into the integral formulas.

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(2) These are all quite similar. Here is a solution for (c).

$x=L$ ——————— $G=0$ here

$\oplus \bullet (x_0, y_0, z_0)$

$x=0$ ——————— $G=0$ here

Sink at $-x_0$, $2L-x_0$

These now generate two images ~~each~~ (sources) at $2L+x_0$ and $-2L+x_0$

Each of these generates an image etc.

$$G = -\frac{1}{4\pi} \sum_{n=-\infty}^{\infty} \left[\frac{1}{|\underline{x}-\underline{y}_n|} - \frac{1}{|\underline{x}-\underline{z}_n|} \right]$$

where $\underline{y}_n = (x_0 + 2Ln, y_0, z_0)$, $\underline{z}_n = (-x_0 + 2Ln, y_0, z_0)$.

3

$$\nabla^2\phi = 0 \qquad x^2+y^2 < R^2$$
$$\phi(R,\theta) = f(\theta)$$

(a) Already know that the solution is

$$\phi(r,\theta) = D_0 + \sum_{n=1}^{\infty}\left(A_n r^n \cos n\theta + B_n r^n \sin n\theta\right)$$

Apply boundary condition at $r=R$ $\Rightarrow$

$$f(\theta) = D_0 + \sum_{n=1}^{\infty} R^n\left(A_n \cos n\theta + B_n \sin n\theta\right)$$

$$D_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta \ , \quad A_n = \frac{R^{-n}}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta$$
$$B_n = \frac{R^{-n}}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\,d\theta$$

$$\phi = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta + \sum_{n=1}^{\infty}\Big[\frac{r^n}{\pi R^n}\left(\int_0^{2\pi} f(\alpha)\cos n\alpha\,d\alpha\right)\cos n\theta$$
$$+ \frac{r^n}{\pi R^n}\left(\int_0^{2\pi} f(\alpha)\sin n\alpha\,d\alpha\right)\sin n\theta\Big]$$

Interchange $\sum$ and $\int$ (uniform convergence)

$$\phi = \frac{1}{\pi}\int_0^{2\pi} f(\alpha)\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{r^n}{R^n}\cos n(\theta-\alpha)\right]d\alpha$$

(b) $$\sum_{n=1}^{\infty}\frac{r^n}{R^n}\cos n(\theta-\alpha) = \mathrm{Real}\left[\sum_{n=1}^{\infty}\left(\frac{re^{i(\theta-\alpha)}}{R}\right)^n\right]$$

This is a geometric series of form $\sum_{n=1}^{\infty} \alpha^n = \frac{\alpha}{1-\alpha}$ for $|\alpha|<1$

Since $\left| \frac{r e^{i(\theta-\alpha)}}{R} \right| < 1$ we have

$$\sum_{1}^{\infty} \frac{r^n}{R^n} \cos n(\theta-\alpha) = \mathrm{Re}\left\{ \frac{r}{R} e^{i(\theta-\alpha)} \frac{1}{1-\frac{r}{R}\cos(\theta-\alpha) - i\frac{r}{R}\sin(\theta-\alpha)} \right\}$$

$$= \mathrm{Re}\left[\frac{r\left(\cos(\theta-\alpha)+i\sin(\theta-\alpha)\right)}{\left(R-r\cos(\theta-\alpha)\right)^2 + r^2\sin^2(\theta-\alpha)} \cdot \left(R - r\cos(\theta-\alpha) + i r \sin(\theta-\alpha)\right) \right]$$

$$= \mathrm{Re}\left[\frac{r\left(R\cos(\theta-\alpha) - r\cos^2(\theta-\alpha) - r\sin^2(\theta-\alpha) + i(\quad)\right)}{R^2 - 2rR\cos(\theta-\alpha) + r^2} \right]$$

$$= \frac{r\left(R\cos(\theta-\alpha) - r\right)}{R^2 - 2rR\cos(\theta-\alpha) + r^2}$$

$$\Rightarrow \quad \frac{1}{2} + \sum_{1}^{\infty} \frac{r^n}{R^n}\cos n(\theta-\alpha) = \frac{1}{2} + (\quad) = \frac{R^2-r^2}{2\left(R^2-2rR\cos(\theta-\alpha)+r^2\right)}$$

as required.

(c) If $f(\theta)=1$, all A_n, B_n are zero

and only D_0 remains; $D_0 = \frac{1}{2\pi}\int_0^{2\pi} 1\, d\theta = 1$.

From the Poisson integral formula

$$\phi(r,\theta) = \frac{R^2-r^2}{2\pi} \int_0^{2\pi} \frac{d\alpha}{R^2 - 2rR\cos(\theta-\alpha) + r^2} \qquad (*)$$

1st put $\theta - \alpha = \xi$ to turn this into

$$\int_0^{2\pi} \frac{d\alpha}{R^2-2rR\cos(\theta-\alpha)+r^2} = \int_{\theta-2\pi}^{\theta} \frac{d\xi}{R^2-2rR\cos(\theta-\alpha)+r^2} = \int_0^{2\pi} \frac{d\xi}{(\quad)}$$

by periodicity.

$$= 2\int_0^{\pi} \frac{d\xi}{R^2-2rR\cos(\xi)+r^2}$$

by symmetry.

Substitution: $t = \tan\frac{\xi}{2}$, casts integral into

$$= 4\int_0^{\infty} \frac{dt}{(R+r)^2t^2+(R-r)^2} = \frac{4}{(R+r)^2}\frac{R+r}{R-r}\left[\tan^{-1}\frac{t(R+r)}{R-r}\right]_0^{\infty}$$

$$= \frac{2\pi}{R^2-r^2}$$

$\Rightarrow \phi(r,\theta) = 1$. from $(*)$

(4) $u_{tt} = c^2u_{xx} \qquad -\infty<x<\infty, \quad t>0$

(a) $(x,t)\to(\xi,\eta)$ $\qquad \xi = x-ct$

$\qquad \eta = x+ct$

$$\frac{\partial}{\partial t} = \frac{\partial\xi}{\partial t}\frac{\partial}{\partial\xi} + \frac{\partial\eta}{\partial t}\frac{\partial}{\partial\eta} = -c\frac{\partial}{\partial\xi}+c\frac{\partial}{\partial\eta}$$

$$\frac{\partial^2}{\partial t^2} = \left(-c\frac{\partial}{\partial\xi}+c\frac{\partial}{\partial\eta}\right)\left(-c\frac{\partial}{\partial\xi}+c\frac{\partial}{\partial\eta}\right) = c^2\frac{\partial^2}{\partial\xi^2} - 2c^2\frac{\partial^2}{\partial\eta\partial\xi} + c^2\frac{\partial^2}{\partial\eta^2}$$

$$\frac{\partial}{\partial x} = \frac{\partial\xi}{\partial x}\frac{\partial}{\partial\xi} + \frac{\partial\eta}{\partial x}\frac{\partial}{\partial\eta}, \quad \frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)^2 = \frac{\partial^2}{\partial\xi^2}+2\frac{\partial^2}{\partial\eta\partial\xi}+\frac{\partial^2}{\partial\eta^2}$$

$\Rightarrow$ PDE becomes $c^2\left(\frac{\partial^2}{\partial\xi^2}-2\frac{\partial^2}{\partial\eta\partial\xi}+\frac{\partial^2}{\partial\eta^2}\right)u = c^2\left(\frac{\partial^2}{\partial\xi^2}+2\frac{\partial^2}{\partial\eta\partial\xi}+\frac{\partial^2}{\partial\eta^2}\right)u$

ie $\quad u_{\xi\eta} = 0$

(b) $u_{\xi\eta} = 0$. Integrate $u_\xi = \alpha_1(\xi)$

Integrate again $u(\xi,\eta) = \alpha(\xi) + \beta(\eta)$

$$u(x,t) = \alpha(x-ct) + \beta(x+ct) \quad \text{as required}$$

(c) Initial conditions

$$u(x,0) = \psi(x) \qquad (1)$$

$$\frac{\partial u}{\partial t}(x,0) = 0 \qquad (2)$$

Using solution from (b) & evaluate at $t=0$ applying (1)-(2)

$$\alpha(x) + \beta(x) = \psi(x)$$

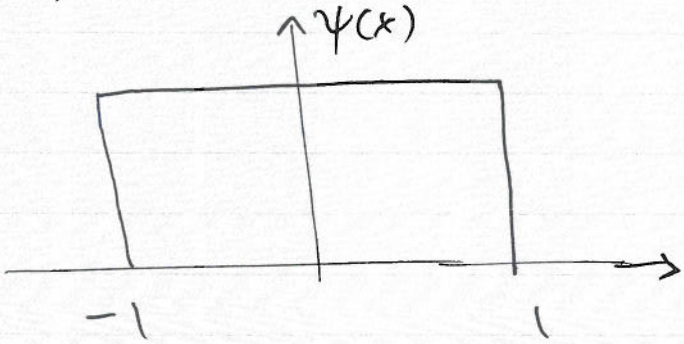
$$-c\alpha'(x) + c\beta'(x) = 0 \Rightarrow \alpha(x) = \beta(x)$$

$$\Rightarrow \alpha(z) = \beta(z) = \tfrac{1}{2}\psi(z)$$

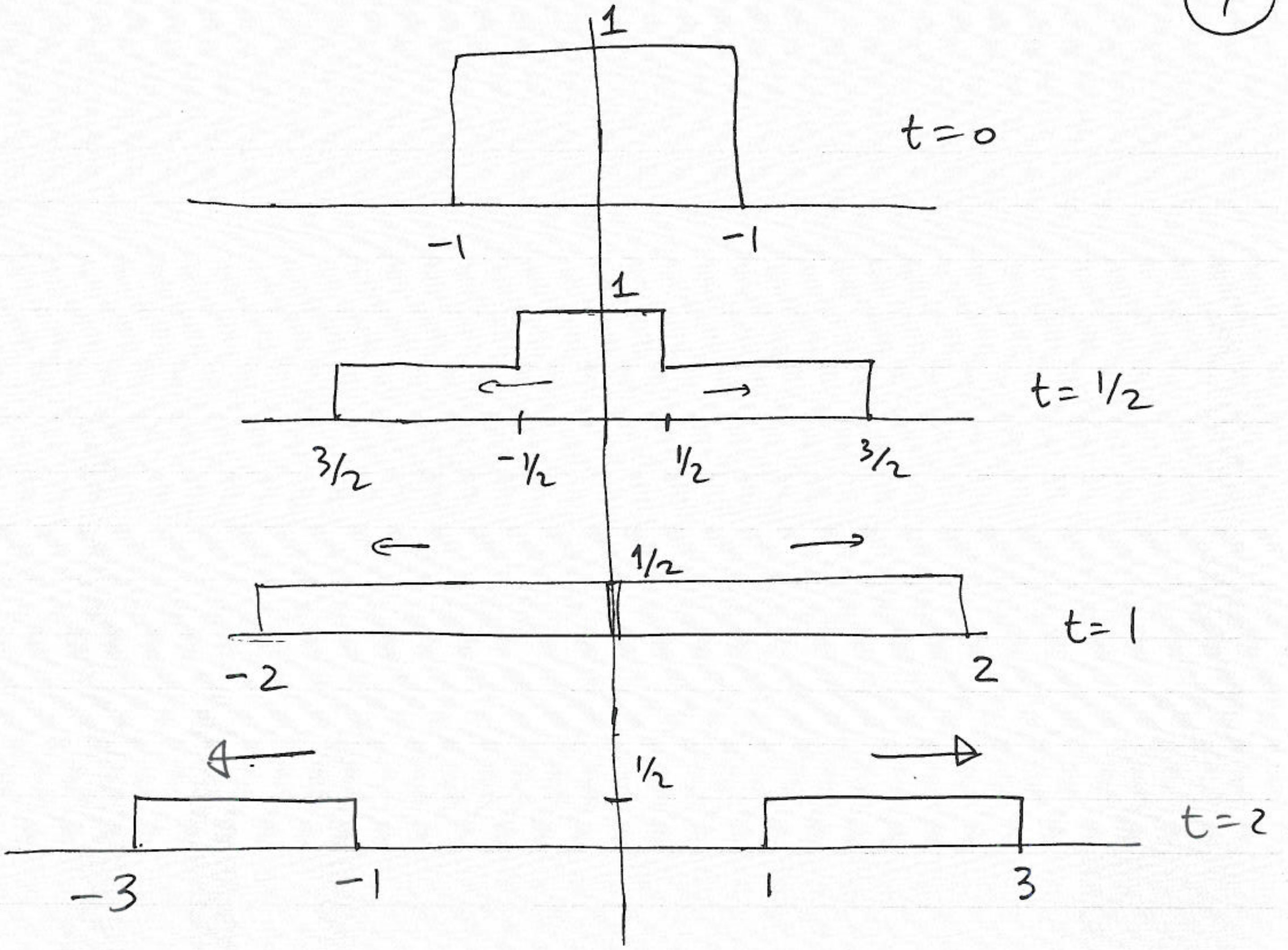
This determines the functional form of α, β. Back to solution in part (b)

$$u(x,t) = \tfrac{1}{2}\psi(x-ct) + \tfrac{1}{2}\psi(x+ct).$$

(d) $c=1$; initial condition is a hat function



$\frac{1}{2}\psi(x-t)$ is half the function above moving to the right with speed 1. $\frac{1}{2}\psi(x+t)$ moves to the left with unit speed.



Pulses move apart.

(5) (a) Spherical polars

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$$\Rightarrow \quad u_{tt} = c^2\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)$$

$$u_{tt} = c^2\frac{1}{r^2}\left(r^2 u_{rr} + 2r u_r\right)$$

$$r u_{tt} = c^2\left(r u_{rr} + 2u_r\right)$$

ie $$(ru)_{tt} = c^2 (ru)_{rr} \qquad 0<r<\infty,\ t>0$$

(b) Let $r u = \psi$ then $\psi_{tt} = c^2 \psi_{rr}$

$\Rightarrow \psi = \big(f(r-ct) + g(r+ct)\big) \Rightarrow$

$u(r,t) = \frac{1}{r}\left[f(r-ct) + g(r+ct)\right]$ as required

6 (a) $u_{xy} = 0 \Rightarrow u(x,y) = f(x) + g(y)$

(b) $u_{xyz} = 0$ Integrate wrt z first.

$u_{xy} = f_1(x,y) \Rightarrow u_x = f_2(x,y) + g_1(x,z)$

$\Rightarrow u = f(x,y) + g(x,z) + h(y,z)$

(c) $u_{xy} = a(x,y)$

$u_x = a_1(x,y) + b_1(x)$

$u = a_2(x,y) + b_2(x) + b_3(y)$

7 Write the equation as

$$\left(\frac{\partial}{\partial x} + 3\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y}\right) = e^{x+y}$$

Want transformation which will make 1st bracket $\frac{\partial}{\partial \xi}$ or $\frac{\partial}{\partial \eta}$ and second bracket the other of $\frac{\partial}{\partial \xi}$ or $\frac{\partial}{\partial \eta}$.

Put $\xi = 3x - y$ $\quad$ ($x = \xi - \eta$, $y = 2\xi - 3\eta$

$\eta = 2x - y$ $\quad$ $\Rightarrow x + y = 3\xi - 4\eta$)

$$\left(\frac{\partial}{\partial x} + 3\frac{\partial}{\partial y}\right) = 3\frac{\partial}{\partial \xi} + 2\frac{\partial}{\partial \eta} - 3\frac{\partial}{\partial \xi} - 3\frac{\partial}{\partial \eta} = -\frac{\partial}{\partial \eta}$$

$$\left(\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y}\right) = 3\frac{\partial}{\partial \xi} + 2\frac{\partial}{\partial \eta} - 2\frac{\partial}{\partial \xi} - 2\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi}$$

$\Rightarrow$ PDE becomes

$$-u_{\xi\eta} = e^{x+y}$$

$$u_{\xi\eta} = -\exp(3\xi - 4\eta)$$
$$= -e^{3\xi} e^{-4\eta}$$

Integrate

$$u_\eta = -\frac{e^{3\xi}}{3} e^{-4\eta} + f_1'(\eta)$$

$$u = \frac{e^{3\xi} e^{-4\eta}}{12} + f_1(\eta) + f_2(\xi)$$

8

$$z^2 = 1 - (x-a)^2 - (y-b)^2$$

Think of $z = z(x,y)$

$$\Rightarrow \quad 2zz_x = -2(x-a) \quad \Rightarrow (x-a)^2 = z^2 z_x^2$$

$$2zz_y = -2(y-b) \quad \Rightarrow (y-b)^2 = z^2 z_y^2$$

So the PDE is

$$z^2 = 1 - z^2(z_x^2 + z_y^2)$$

or $z^2(1 + z_x^2 + z_y^2) = 1$

9 $$u_x^2 + u_y^2 = 1$$

try $u = f(x) + g(y)$

$$u_x = f'(x) \qquad u_y = g'(y)$$

$$\Rightarrow \quad f'^2(x) + g'^2(y) = 1$$

$$\underbrace{(f'(x))^2}_{\text{fn of } x} = \underbrace{1 - (g'(y))^2}_{\text{fn of } y}$$

$\Rightarrow$ must be const., λ say.

$$(f'(x))^2 = \lambda^2 \qquad f' = \pm\lambda \qquad \underline{f = \pm\lambda x}$$

$$1 - (g'(y))^2 = \lambda^2 \qquad g'^2 = 1 - \lambda^2 \qquad |\lambda| < 1$$

$$\Rightarrow \quad \underline{g(y) = \pm\sqrt{1-\lambda^2}\, y}$$

10 $$u_x u_y = 1$$

Try $u = f(x) + g(y)$

Then $f'(x)\, g'(y) = 1$.

ie $f'(x) = \dfrac{1}{g'(y)} = \lambda$ a constant.

$$f(x) = \lambda x + \lambda_1 \qquad g(y) = \frac{1}{\lambda} y + \lambda_2$$

Solutions of the form $u = f(x)g(y)$

$$f'(x)g(y) \cdot f(x)g'(y) = 1$$

$$f(x)f'(x) = \frac{1}{g(y)g'(y)} = \lambda \quad \text{const.}$$

$$\Rightarrow \quad ff' = \lambda \Rightarrow \frac{1}{2}f^2(x) = \lambda x + \lambda_0$$

$$f(x) = \pm\sqrt{2(\lambda x + \lambda_0)}$$

Similarly $\quad g(y) = \pm\sqrt{2\left(\frac{y}{\lambda} + \lambda_1\right)}$