M2AA2 - MULTIVARIABLE CALCULUS

SOLUTIONS - PROBLEM SHEET 6

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1. $$I = \int_S \frac{\underline{x}\cdot\underline{n}}{|\underline{x}|^3}\, dS$$

If $\underline{F} = \dfrac{\underline{x}}{|\underline{x}|^3} = \dfrac{(x,y,z)}{(x^2+y^2+z^2)^{3/2}} \Rightarrow \nabla\cdot\underline{F} = 0$ if $\underline{x} \neq 0$

On the sphere $|\underline{x}| = R$, $\underline{n} = \dfrac{(x,y,z)}{R}$

$$\Rightarrow I = \int_{\text{Sphere}} \frac{(x^2+y^2+z^2)}{R^4}\, dS = \frac{1}{R^2}\int_{\text{Sphere}} dS = \frac{4\pi R^2}{R^2} = 4\pi$$

If $\underline{S}$ does not bound the origin, then

$$\int_S \frac{\underline{x}\cdot\underline{n}}{|\underline{x}|^3}\, dS = \int_V \nabla\cdot\left(\frac{\underline{x}}{|\underline{x}|^3}\right) dV = 0.$$

Using these facts, we see that if $\underline{E} = \dfrac{q}{4\pi\epsilon_0}\dfrac{\underline{x}-\underline{a}}{|\underline{x}-\underline{a}|^3}$

we immediately obtain the desired result by shifting the origin to $\underline{x} = \underline{a}$.

2. $$\underline{F}(\underline{x}) = r^{-1}\, \underline{e}_\theta$$

$$\text{curl}\, \underline{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\underline{e}_1 & h_2\underline{e}_2 & h_3\underline{e}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix} = \frac{1}{r}\begin{vmatrix} \underline{e}_r & r\underline{e}_\theta & \underline{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & 1 & 0 \end{vmatrix}$$

$$= \underline{0}$$

$$\oint \underline{F}\cdot d\underline{s} = \oint \underline{e}_\theta \cdot d\underline{s} \quad \text{since we are on } r=1$$

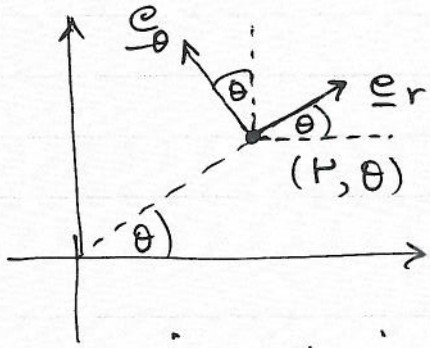
$$d\underline{s} = (dx, dy) = \big(d(\cos\theta), d(\sin\theta)\big) = (-\sin\theta, \cos\theta)\,d\theta = \underline{e}_\theta \, d\theta$$

$\Rightarrow \oint \underline{F}\cdot d\underline{s} = \oint d\theta = 2\pi$, but Stokes's theorem which is

$$\iint_S \nabla\times\underline{F}\, dS = \oint_C \underline{F}\cdot d\underline{s} \quad \text{does not hold.}$$

The reason is that $\underline{F}$ is not continuous in the region under consideration; $\underline{F}$ is singular at $r=0$.

③



In terms of $\underline{i}$ and $\underline{j}$ we have

$$\underline{e}_r = \cos\theta\,\underline{i} + \sin\theta\,\underline{j}, \quad \underline{e}_\theta = -\sin\theta\,\underline{i} + \cos\theta\,\underline{j}$$

$$\frac{\partial \underline{e}_r}{\partial\theta} = -\sin\theta\,\underline{i} + \cos\theta\,\underline{j} = \underline{e}_\theta \qquad \frac{\partial \underline{e}_r}{\partial r} = \frac{\partial \underline{e}_\theta}{\partial r} = 0.$$

$$\frac{\partial \underline{e}_\theta}{\partial\theta} = -\cos\theta\,\underline{i} - \sin\theta\,\underline{j} = -\underline{e}_r \qquad \frac{\partial \underline{e}_z}{\partial z} = \frac{\partial \underline{e}_z}{\partial r} = \frac{\partial \underline{e}_z}{\partial \theta} = 0$$

Given $\nabla = \underline{e}_r \dfrac{\partial}{\partial r} + \underline{e}_\theta \dfrac{1}{r}\dfrac{\partial}{\partial\theta} + \underline{e}_z \dfrac{\partial}{\partial z}$

$$\begin{aligned}
\nabla\cdot\underline{A} = {} & \underline{e}_r \cdot \frac{\partial}{\partial r}\left(A_1\underline{e}_r + A_2\underline{e}_\theta + A_3\underline{e}_z\right) \\
& + \underline{e}_\theta \cdot \frac{1}{r}\frac{\partial}{\partial\theta}\left(A_1\underline{e}_r + A_2\underline{e}_\theta + A_3\underline{e}_z\right) \\
& + \underline{e}_z \cdot \frac{\partial}{\partial z}\left(A_1\underline{e}_r + A_2\underline{e}_\theta + A_3\underline{e}_z\right)
\end{aligned}$$

$$= \underline{e}_r \cdot \left(\frac{\partial A_1}{\partial r}\underline{e}_r + \frac{\partial A_2}{\partial r}\underline{e}_\theta + \frac{\partial A_3}{\partial z}\underline{e}_z \right)$$

$$+ \underline{e}_\theta \frac{1}{r} \left(\frac{\partial A_1}{\partial \theta}\underline{e}_r + A_1 \frac{\partial \underline{e}_r}{\partial \theta} + \frac{\partial A_2}{\partial \theta}\underline{e}_\theta + A_2 \frac{\partial \underline{e}_\theta}{\partial \theta} + \frac{\partial A_3}{\partial \theta}\underline{e}_z \right)$$

$$+ \underline{e}_z \cdot \left(\frac{\partial A_1}{\partial z}\underline{e}_r + \frac{\partial A_2}{\partial z}\underline{e}_\theta + \frac{\partial A_3}{\partial z}\underline{e}_z \right)$$

$$= \underline{e}_r \cdot \left(\frac{\partial A_1}{\partial r}\underline{e}_r \right) + \underline{e}_\theta \frac{1}{r} \cdot \left(A_1 \underline{e}_\theta + \frac{\partial A_2}{\partial \theta}\underline{e}_\theta \right) + \underline{e}_z \cdot \left(\frac{\partial A_3}{\partial z}\underline{e}_z \right)$$

$$= \frac{\partial A_1}{\partial r} + \frac{1}{r}A_1 + \frac{1}{r}\frac{\partial A_2}{\partial \theta} + \frac{\partial A_3}{\partial z}$$

which is the familiar formula

$$\nabla \times \underline{A} = \underline{e}_r \times \left[\frac{\partial A_1}{\partial r}\underline{e}_r + \frac{\partial A_2}{\partial r}\underline{e}_\theta + \frac{\partial A_3}{\partial r}\underline{e}_z \right]$$

$$+ \underline{e}_\theta \frac{1}{r} \times \left[\frac{\partial A_1}{\partial \theta}\underline{e}_r + A_1\underline{e}_\theta + \frac{\partial A_2}{\partial \theta}\underline{e}_\theta - A_2\underline{e}_r + \frac{\partial A_3}{\partial \theta}\underline{e}_z \right]$$

$$+ \underline{e}_z \times \left[\frac{\partial A_1}{\partial z}\underline{e}_r + \frac{\partial A_2}{\partial z}\underline{e}_\theta + \frac{\partial A_3}{\partial z}\underline{e}_z \right]$$

$$= \frac{\partial A_2}{\partial r}(\underline{e}_r \times \underline{e}_\theta) + \frac{\partial A_3}{\partial r}(\underline{e}_r \times \underline{e}_z)$$

$$+ \frac{1}{r}\left(\frac{\partial A_1}{\partial \theta} - A_2 \right)(\underline{e}_\theta \times \underline{e}_r) + \frac{1}{r}\frac{\partial A_3}{\partial \theta}(\underline{e}_\theta \times \underline{e}_z)$$

$$+ \frac{\partial A_1}{\partial z}(\underline{e}_z \times \underline{e}_r) + \frac{\partial A_2}{\partial z}(\underline{e}_z \times \underline{e}_\theta)$$

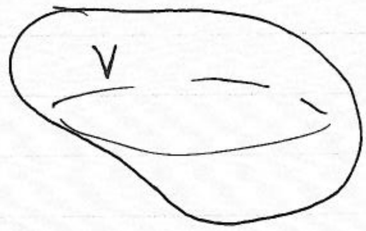
Now use RH rule for cross products.

$$\underline{e}_r \times \underline{e}_\theta = \underline{e}_z \qquad \underline{e}_r \times \underline{e}_z = -\underline{e}_\theta$$

$$\underline{e}_\theta \times \underline{e}_r = -\underline{e}_z \qquad \underline{e}_\theta \times \underline{e}_z = \underline{e}_r$$

$$\nabla \times \underline{A} = \left(\frac{1}{r}\frac{\partial A_3}{\partial \theta} - \frac{\partial A_2}{\partial z} \right)\underline{e}_r + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial r} \right)\underline{e}_\theta + \left(\frac{\partial A_2}{\partial r} + \frac{1}{r}A_2 - \frac{1}{r}\frac{\partial A_1}{\partial \theta} \right)\underline{e}$$

(4)



$\partial V = S$ $\quad \nabla^2 u = m^2 u \quad \underline{x} \in \mathbb{R}^3$

u, or $\frac{\partial u}{\partial n}$ given.

Let u_1, u_2 be two such solutions.

Then $U = u_1 - u_2$ satisfies $\quad \nabla^2 U = m^2 U$ in V

U or $\frac{\partial U}{\partial n} = 0$ on S

Consider $$\int_V \nabla\cdot(U\nabla U)\,dV = \int_V \left(U\nabla^2 U + |\nabla U|^2\right)dV = \int_V \underbrace{\left(m^2 U^2 + |\nabla U|^2\right) dV}_{\geq 0}$$

But using the Divergence Thm, we also have

$$\int_V \nabla\cdot(U\nabla U)\,dV = \int_S U\frac{\partial U}{\partial n}\,dS = 0$$ since U or $\frac{\partial U}{\partial n}$ are zero on S.

$$\Rightarrow \int_V \left(m^2U^2 + |\nabla U|^2\right)dV = 0 \Rightarrow U \equiv 0$$ ie <u>uniqueness</u>

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(5) First part is in your notes, ie

$$\phi(r,\theta) = C_0 \ln r + D_0 + \sum_{n=1}^{\infty}\Big[\left(A_n r^n + B_n r^{-n}\right)\cos n\theta + \left(C_n r^n + D_n r^{-n}\right)\sin n\theta\Big]$$

To solve $\quad \nabla^2\phi = 0 \quad r < a$

$\phi(a,\theta) = \sin\theta$

Bounded $\phi \Rightarrow$ $$\phi(r,\theta) = D_0 + \sum_{n=1}^{\infty}\left[C_n r^n \sin n\theta + A_n r^n \cos n\theta\right]$$

At $\underline{r=a}$ $\quad \sin\theta = D_0 + \sum_{n=1}^{\infty} [C_n a^n \sin n\theta + A_n a^n \cos n\theta]$

Hence $D_0 = A_n = 0$, $C_1 = \frac{1}{a}$, $C_2 = C_3 = \cdots = 0$

$$\phi(r,\theta) = \frac{r}{a}\sin\theta$$

$\underline{r>a}$ $\quad \phi \to 0$ as $r \to \infty$ $\Rightarrow$

$$\phi(r,\theta) = \sum_{n=1}^{\infty} [B_n r^{-n} \cos n\theta + D_n r^{-n} \sin n\theta]$$

Again BC at $r=a$ gives $B_n = 0$ $\quad D_1 = a$
$D_2 = D_3 = \cdots = 0$

$$\Rightarrow \quad \phi(r,\theta) = \frac{a}{r}\sin\theta .$$

6. $\phi = \phi(r)$ $\qquad r = (x^2+y^2+z^2)^{1/2}$ $\qquad \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$ etc

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = \left(\frac{\partial\phi}{\partial r}\frac{\partial r}{\partial x}, \frac{\partial\phi}{\partial r}\frac{\partial r}{\partial y}, \frac{\partial\phi}{\partial r}\frac{\partial r}{\partial z}\right)$$

$$= \phi'(r)\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) = \phi'(r)\,\frac{\underline{x}}{r}$$

$$\nabla^2\phi = \nabla\cdot(\nabla\phi) = \frac{\partial}{\partial x}\left(\phi'(r)\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\phi'(r)\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(\phi'(r)\frac{z}{r}\right)$$

$$= \frac{\phi'(r)}{r} - \frac{x^2}{r^3}\phi'(r) + \phi''(r)\frac{x}{r} + 6 \text{ similar terms}$$

$$= \phi''(r) + \frac{2}{r}\phi'(r) \qquad \text{– see notes also}$$

If $\nabla^2\phi = 1$ and $\phi(a) = 1$ $\quad 0<r<a$

Since BC is independent of θ, $\phi \equiv \phi(r) \Rightarrow$

$$\phi'' + \frac{2}{r}\phi' = 1 \Rightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2\phi'\right) = 1$$

$$r^2\phi' = \frac{r^3}{3} + A \rightarrow \phi = \frac{r^2}{6} - \frac{A}{r} + B$$

$A = 0$ for finite solutions at $r = 0$, $\phi(a) = 1 \Rightarrow \underline{\frac{a^2}{6} + B = 1}$

$$\phi(r) = \frac{1}{6}(r^2 - a^2) + 1$$

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$$\nabla^2\phi = f \quad \text{in } V$$

$$g\frac{\partial\phi}{\partial n} + \phi = 0 \quad \text{on } S \qquad g(\underline{x}) \geq 0 \text{ on } S.$$

Suppose there exist two solutions ϕ_1, ϕ_2 and consider the problem for the difference $\Phi = \phi_1 - \phi_2$
We have

$$\nabla^2\Phi = 0 \quad \text{in } V$$

$$g\frac{\partial\Phi}{\partial n} + \Phi = 0 \quad \text{on } S. \qquad (*)$$

Now $\int_V \nabla\cdot(\Phi\nabla\Phi)\,dV = \int_V \left[\Phi\nabla^2\Phi + |\nabla\Phi|^2\right]dV = \int_V |\nabla\Phi|^2\,dV$

Also $\int_V \nabla\cdot(\Phi\nabla\Phi)\,dV = \int_S \Phi\frac{\partial\Phi}{\partial n}\,dS = \int_S -g(\underline{x})\left(\frac{\partial\Phi}{\partial n}\right)^2 dS$
using $(*)$

Hence $\int_V |\nabla\Phi|^2\,dV = \int -g(\underline{x})\left(\frac{\partial\Phi}{\partial n}\right)^2 dS \leq 0$ by hypothesis

Can only be satisfied if $\Phi = \text{const.}$

But the const. $= 0$ on $S \Rightarrow \Phi = 0 \Rightarrow \underline{\phi_1 = \phi_2}$

If $g(\underline{x}) < 0$ theorem cannot be proven as above.

$\phi = x$ – clearly $\nabla^2\phi = 0$

In spherical polars, $\frac{\partial\phi}{\partial n} = \nabla\phi \cdot \underline{n} = \underline{i} \cdot (\sin\theta\cos\varphi, \cdot, \cdot)$

$= \sin\theta\cos\varphi = x$

$\Rightarrow$ If $g = -1$ we have a solution.

satisfying $g\frac{\partial\phi}{\partial n} + \phi = 0$

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$$\int_V |\nabla(w-u)|^2 dV = \int_V (\nabla w - \nabla u) \cdot (\nabla w - \nabla u)\, dV$$

$$= \int_V \left(|\nabla w|^2 - 2\nabla w \cdot \nabla u + |\nabla u|^2\right) dV = \int |\nabla w|^2 dV + \int |\nabla u|^2 dV - \int 2(\nabla w \cdot \nabla u)\, dV$$

Therefore

$$\int_V |\nabla(w-u)|^2 dV + 2\int_V \nabla u \cdot (\nabla(w-u))\, dV$$

$$= \int |\nabla w|^2 dV - \int_V |\nabla u|^2 dV \quad \text{as required.}$$

We need to prove that the LHS ≥ 0. Clearly 1st term is

for $\int_V \nabla u \cdot \nabla(w-u)\, dV$ consider $\int_V \nabla \cdot [(w-u)\nabla u]\, dV$

$$\Rightarrow \int_V \nabla \cdot [(w-u)\nabla u]\, dV = \int_V (w-u)\nabla^2 u\, dV + \int_V \nabla(w-u) \cdot \nabla u\, dV$$

Divergence thm. $\rightarrow \int_S (w-u)\frac{\partial u}{\partial n} dS = 0$ since $u = w$ on S

Also $\nabla^2 u = 0$ in V $\Rightarrow \int_V \nabla u \cdot \nabla(w-u)\, dV = 0$

This completes the proof.

9 for harmonic functions ϕ, ψ, $\nabla^2\phi = 0$, $\nabla^2\psi = 0$ we have $\nabla\cdot(\psi\nabla\phi) - \nabla\cdot(\phi\nabla\psi) = 0$; by the divergence theorem we get

$$0 = \int_V \nabla\cdot(\psi\nabla\phi - \phi\nabla\psi)\,dV = \int_S \left(\psi\frac{\partial\phi}{\partial n} - \phi\frac{\partial\psi}{\partial n}\right)dS$$

Now $\psi = \frac{1}{r}$ is harmonic if $r \neq 0 \Rightarrow$ result follows.

If V is the volume $\varepsilon \le r \le a$, we have two surfaces $S_1 : r = a$, $S_\varepsilon : r = \varepsilon$. We have

$$\int_{S_1+S_\varepsilon} \left(\phi\nabla(\tfrac{1}{r}) - \tfrac{1}{r}\nabla\phi\right)\cdot \underline{n}\,dS = 0 \quad \Rightarrow$$

On S_1: $\displaystyle\int_{S_1}\left(\phi(-\tfrac{1}{r^2})\hat{\underline{r}} - \tfrac{1}{r}\nabla\phi\right)\cdot\hat{\underline{r}}\,dS = -\frac{1}{a^2}\int_{S_1}\phi\,dS - \frac{1}{a}\int_{S_1}\nabla\phi\cdot\underline{n}\,dS$

Now ϕ is harmonic $\Rightarrow \displaystyle\int_V \nabla\cdot(\nabla\phi)\,dV = \int_V \nabla^2\phi\,dV = 0$

and $\displaystyle\int_V \nabla\cdot(\nabla\phi)\,dV = \int_{S_1}\nabla\phi\cdot\underline{n}\,dS = 0.$

So $$\int_{S_1}\left[\phi\nabla(\tfrac{1}{r}) - \tfrac{1}{r}\nabla\phi\right]\cdot\underline{n}\,dS = -\frac{1}{a^2}\int_{S_1}\phi\,dS \quad (*)$$

Now consider S_ε. Here $\underline{n} = -\hat{\underline{r}}$

$$\int_{S_\varepsilon}(\;) = \int_{r=\varepsilon}\phi\left(-\tfrac{1}{\varepsilon^2}\right)\hat{\underline{r}}\cdot(-\hat{\underline{r}})\,dS + \int_{r=\varepsilon}\left(+\tfrac{1}{\varepsilon}\nabla\phi\right)\cdot\hat{\underline{r}}\,dS$$

$\int_{S_\varepsilon} \to \phi(0)\, 4\pi + O(\varepsilon)$. Put this together with (*)

to give $\phi(0) = \frac{1}{4\pi a^2} \int_{S_1} \phi \, dS$ as required.

To prove the maximum principle using this.

Suppose ϕ attains a maximum at an interior point. Let this value be M and move the origin so that the point where the max is attained is the origin.

Then

$$\phi(0) = \frac{1}{4\pi a^2} \int_{S_1} \phi \, dS$$

, so for a small enough we have

$$\phi(0) = M = \frac{1}{4\pi a^2} \int_{S_1} \phi \, dS \le \frac{1}{4\pi a^2} \int_{S_1} M \, dS = M$$

ie $M \le M$ for any sphere radius a which is contained in V.

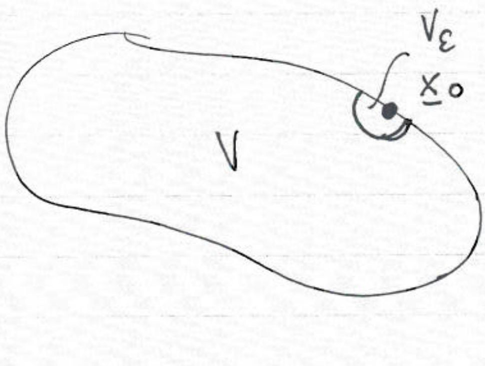
$\Rightarrow$ $\phi = M$ for every sphere around the origin

$\Rightarrow$ ϕ is a constant. Assumption of a local maximum invalid. $\Rightarrow$ maximum has to be on the boundary.

For a minimum $\phi \to -\phi$ and repeat argument.

(10) First part is in your notes. Identify $G(\underline{x}; \underline{x}_0) = -\frac{1}{4\pi |\underline{x} - \underline{x}_0|}$

If $\underline{x}_0$ is on the boundary of S



puncture a semi-circular sphere out.

Basic formula comes from

$$\iiint_{V-V_\varepsilon} \left(\phi \nabla^2 G - G \nabla^2 \phi\right) dV = \iint_{S+S_\varepsilon} \left(\phi \frac{\partial G}{\partial n} - G \frac{\partial \phi}{\partial n}\right) dS$$

where $\nabla^2 \phi = f(x)$ in V

$\nabla^2 G = 0$ in $V - V_\varepsilon$

The LHS $\longrightarrow \displaystyle\int_V -G f \, dV = \frac{1}{4\pi} \int_V \frac{f(\underline{x})}{|\underline{x} - \underline{x}_0|} dV$

$$\text{RHS} \rightarrow \int_S \left[\phi \frac{\partial}{\partial n}\left(-\frac{1}{4\pi |\underline{x} - \underline{x}_0|}\right) + \frac{1}{4\pi |\underline{x} - \underline{x}_0|} \frac{\partial \phi}{\partial n}\right] dS$$

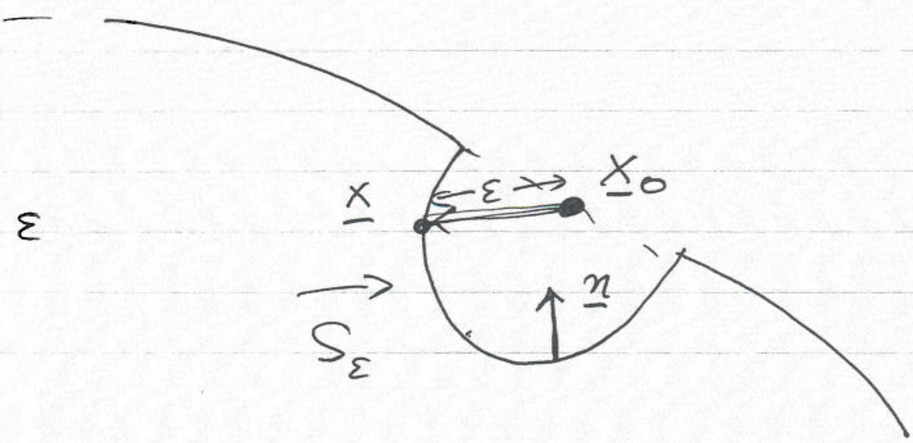
$$= \frac{1}{4\pi} \int_S \left(\frac{1}{|\underline{x} - \underline{x}_0|} \frac{\partial \phi}{\partial n}(\underline{x}) - \phi(\underline{x}) \frac{\partial}{\partial n}\left(\frac{1}{|\underline{x} - \underline{x}_0|}\right)\right) dS$$

So all the details are in $I_\varepsilon = \displaystyle\int_{S_\varepsilon} (\ . \ )$

$$I_\varepsilon = \frac{1}{4\pi} \int_{S_\varepsilon} \left[\frac{1}{|\underline{x} - \underline{x}_0|} \frac{\partial \phi}{\partial n} - \phi(\underline{x}) \frac{\partial}{\partial n}\left(\frac{1}{|\underline{x} - \underline{x}_0|}\right)\right] dS$$

Introduce a spherical polars coordinate system with origin at $\underline{x} = \underline{x}_0$

$S_\varepsilon : |\underline{x} - \underline{x}_0| = \varepsilon$



$$I_\varepsilon \approx \frac{1}{4\pi} \int_{S_\varepsilon} \left[\frac{1}{\varepsilon} \frac{\partial \phi}{\partial n}(\underline{x}_0) - \phi(\underline{x}_0) \left(-\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right)_{r=\varepsilon} \right] dS$$

$$\approx \frac{1}{4\pi} \int_{S_\varepsilon} \left[\frac{1}{\varepsilon} \frac{\partial \phi}{\partial n}(\underline{x}_0) - \frac{\phi(\underline{x}_0)}{\varepsilon^2} \right] dS$$

$$\longrightarrow -\frac{\phi(\underline{x}_0)}{4\pi\varepsilon^2} \int_{S_\varepsilon} dS = -\frac{\phi(\underline{x}_0)}{4\pi\varepsilon^2} \cdot 2\pi\varepsilon^2$$

$$= -\frac{1}{2}\phi(\underline{x}_0)$$

<u>Note</u> The $\int_{S_\varepsilon} dS = 2\pi\varepsilon^2$ because it is <u>half a sphere</u>.

Put together to find

$$4\pi\left(\frac{1}{2}\phi(\underline{x}_0)\right) = -\int_V \frac{f(\underline{x})}{|\underline{x}-\underline{x}_0|} dV + \int_S \left[\frac{1}{|\underline{x}-\underline{x}_0|} \frac{\partial \phi}{\partial n}(\underline{x}) - \phi(\underline{x}) \frac{\partial}{\partial n}\left(\frac{1}{|\underline{x}-\underline{x}_0|}\right) \right] dS$$

If $\underline{x}_0$ is inside V, ie not on the Boundary then the LHS is with $1\,\phi(\underline{x}_0)$ not $\frac{1}{2}$.