M2AA2 - MULTIVARIABLE CALCULUS

PROBLEM SHEET 5. PROF. D.T. PAPAGEORGIOU

1. Use
$$\delta \underline{x} = h_1 \delta u \, \underline{e}_1 + h_2 \delta v \, \underline{e}_2 + h_3 \delta z \, \underline{e}_3$$
$$= \delta x_1 \, \underline{i} + \delta x_2 \, \underline{j} + \delta x_3 \, \underline{k}$$

$$\delta x_1 = u \delta u - v \delta v, \quad \delta x_2 = v \delta u + u \delta v \quad \delta x_3 = \delta z$$

$$\Rightarrow \delta \underline{x} = (u \underline{i} + v \underline{j}) \delta u + (u \underline{j} - v \underline{i}) \delta v + \delta z \, \underline{k}$$

So $h_1 \underline{e}_1 = (u \underline{i} + v \underline{j}) \quad h_2 \underline{e}_2 = (u \underline{j} - v \underline{i}) \quad \underline{h_3 = 1 \;\; \underline{e}_3 = \underline{k}}$

$$\Rightarrow h_1^2 = u^2 + v^2, \quad h_2^2 = u^2 + v^2 \quad \text{ie} \quad \underline{h_1 = h_2 = \sqrt{u^2 + v^2}}$$

and
$$\underline{e}_1 = \frac{1}{\sqrt{u^2+v^2}} (u \underline{i} + v \underline{j})$$

$$\underline{e}_2 = \frac{1}{\sqrt{u^2+v^2}} (-v \underline{i} + u \underline{j})$$

2.
$$\underline{F} = u(u^2+v^2)^{3/2} \underline{e}_1 - v(u^2+v^2)^{3/2} \underline{e}_2 \stackrel{\text{def}}{=} F_1 \underline{e}_1 + F_2 \underline{e}_2$$

(a)
$$\nabla \cdot \underline{F} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u}(h_2 h_3 F_1) + \frac{\partial}{\partial v}(h_1 h_3 F_2) \right]$$
$$= \frac{1}{u^2+v^2} \left[\frac{\partial}{\partial u}\left(u(u^2+v^2)^2\right) - \frac{\partial}{\partial v}\left(v(u^2+v^2)^2\right) \right]$$
$$= \frac{1}{u^2+v^2} \left[(u^2+v^2)^2 + 4u^2(u^2+v^2) - (u^2+v^2)^2 - 4v^2(u^2+v^2) \right]$$
$$= \underline{4(u^2 - v^2)}$$

(b) Note $F_3 \equiv 0$ and $\underline{F}$ is independent of z, so only 3rd component of $\nabla \times \underline{F}$ is non-zero, ie

$$\frac{\underline{e}_3}{h_1 h_2 h_3}\left[-\frac{\partial}{\partial u}\left(v(u^2+v^2)^2\right)-\frac{\partial}{\partial v}\left(u(u^2+v^2)^2\right)\right] = -8uv\,\underline{e}_3 .$$

(c) $$\underline{F} = F_1\,\underline{e}_1 + F_2\,\underline{e}_2 = u(u^2+v^2)^{3/2}\left[\frac{u}{\sqrt{u^2+v^2}}\,\underline{i} + \frac{v}{\sqrt{u^2+v^2}}\,\underline{j}\right]$$
$$- v(u^2+v^2)^{3/2}\left[-\frac{v}{\sqrt{u^2+v^2}}\,\underline{i} + \frac{u}{\sqrt{u^2+v^2}}\,\underline{j}\right]$$

where the forms of $\underline{e}_1$, $\underline{e}_2$ in terms of $\underline{i}$, $\underline{j}$ come from Problem 1.

Simplify to
$$\underline{F} = (u^2+v^2)^2\,\underline{i} = 4(x_1^2+x_2^2)\,\underline{i}$$

$$\Rightarrow \quad \nabla\cdot\underline{F} = 8x_1 = 4(u^2-v^2) \quad \text{as before}$$

$$\nabla\times\underline{F} = -\frac{\partial}{\partial x_2}\left[4x_1^2+4x_2^2\right]\underline{k} = -8x_2\,\underline{k} = -8uv\,\underline{e}_3$$
as before

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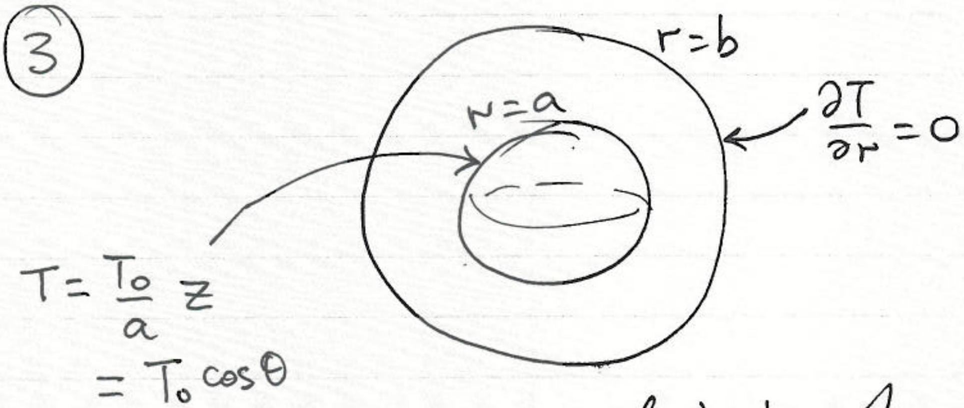

Use spherical polars (r, θ, φ)

We know that $r\cos\theta$ and $r^{-2}\cos\theta$ are solutions of Laplace's equation. Hence

$$T(r,\theta,\varphi) = A\,r\cos\theta + B\,r^{-2}\cos\theta \quad \text{is also a solution.}$$

Expect solution to be independent of φ since the BCs are themselves ind. of φ.

To find A, B we need to satisfy the BCs.

(i) $r=a$ $\quad A a\cos\theta + \dfrac{B}{a^2}\cos\theta = T_0\cos\theta$ $\quad$ (1)

(ii) $r=b$ $\quad A\cos\theta - 2\dfrac{B}{b^3}\cos\theta = 0$ $\quad$ (2)

(2) $\Rightarrow$ $\quad A = 2\dfrac{B}{b^3}$

Therefor (1) becomes $\quad \left(2\dfrac{a}{b^3} + \dfrac{1}{a^2}\right)B = T_0$

$$\Rightarrow \quad B = \frac{a^2 b^3 T_0}{2a^3+b^3}, \quad A = \frac{2T_0 a^2}{2a^3+b^3}$$

(4) To prove uniqueness of the solution in (3) we assume that there are 2 solutions T_1, T_2 satisfying the same problem, ie

$$\nabla^2 T_1 = 0 \qquad\qquad \nabla^2 T_2 = 0$$

$$T_1 = T_0\cos\theta, \ r=a \qquad\qquad T_2 = T_0\cos\theta \quad r=a$$

$$\frac{\partial T_1}{\partial r} = 0, \quad r=b \qquad\qquad \frac{\partial T_2}{\partial r} = 0 \quad \text{on } r=b$$

So subtracting & letting $T = T_1 - T_2$, we have for the problem for T,

$$\nabla^2 T = 0$$

$$T = 0 \quad \text{on } r=a$$

$$\frac{\partial T}{\partial r} = 0 \quad \text{on } r=b$$

Now $\int_V \nabla\cdot(T\nabla T)\,dV = \int_V \left(T\nabla^2 T + |\nabla T|^2\right)dV$

$= \int_S T\nabla T\cdot \underline{n}\; dS$

the last integral coming from the divergence theorem.

Now $\nabla T\cdot\underline{n} = \pm\frac{\partial T}{\partial r}$ (+ on $r=b$, − on $r=a$)

but in any case is zero ~~since~~ at $r=b$.

At $r=a$, $T=0$, therefore $\int_S T\nabla T\cdot\underline{n}\,dS = 0$

Also $\nabla^2 T = 0$, hence $\int_V |\nabla T|^2 dV = 0$ which implies $\nabla T = 0$, ie $T=$ const. The constant is 0, since $T=0$ on $r=a$, hence the uniqueness result.

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(5) Since $\underline{B}$ is parallel to the family of $f(\underline{x})=$const.

this means that $\underline{B} = \alpha\nabla f$ for some α (const).

$\Rightarrow \underline{B}\cdot(\nabla\times\underline{B}) = \alpha\nabla f\cdot(\nabla\times\alpha\nabla f)$

$= \alpha^2\nabla f\cdot(\nabla\times\nabla f)$

But $\nabla\times\nabla f = 0$, hence the result.

(6.) Ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{1} - 1 = 0$

ie $G(x,y,z)=0$ $\qquad \underline{n} = \nabla G/|\nabla G|$

$$\underline{n} = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, 2z\right)\frac{1}{\sqrt{\frac{4x^2}{a^4}+\frac{4y^2}{b^4}+4z^2}} = \left(\frac{x}{a^2}, \frac{y}{b^2}, z\right)\frac{1}{\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+z^2}}$$

and of course we are on the surface ie $z^2 = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$.

For $\underline{F} = (-y, x, 0)$ $\quad \underline{F} \cdot \underline{n} = \left(-\frac{xy}{a^2} + \frac{xy}{b^2}\right) \frac{1}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + z^2}}$

So if $a = b$, $\int_S \underline{F} \cdot \underline{n} \, dS = 0$.

If $a \neq b$ we need to compute

$$I = \int_S \frac{a^2 - b^2}{a^2 b^2} xy \frac{1}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + z^2}} dS$$

We have $\quad dS \, \underline{n} \cdot \hat{\underline{z}} = dx dy$ (see notes)

ie $dS \frac{z}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + z^2}} = dx dy$, hence

$$I = \iint_{S_p} \frac{a^2 - b^2}{a^2 b^2} \frac{xy \, dx dy}{z} = \frac{a^2 - b^2}{a^2 b^2} \iint_{S_p} \frac{xy}{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}} dx dy$$

where S_p is the projection of S onto the (x, y)-plane

Now we are told that the projection is the unit square, ie $0 \leq x \leq 1$, $0 \leq y \leq 1$. In fact, that is how the surface S is defined. Hence the integral becomes

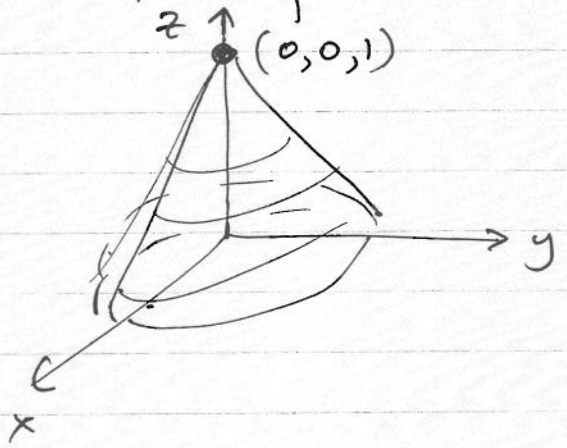
$$I = \frac{a^2 - b^2}{a^2 b^2} \int_0^1 \int_0^1 \frac{xy \, dx dy}{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}} = \frac{a^2 - b^2}{a^2 b^2} \int_0^1 -a^2 \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) \Bigg|_0^1 y \, dy$$

$$= \frac{a^2 - b^2}{a^2 b^2} \int_0^1 a^2 y \left[\left(1 - \frac{y^2}{b^2}\right)^{1/2} - \left(1 - \frac{1}{a^2} - \frac{y^2}{b^2}\right)^{1/2}\right] dy$$

$$= \frac{a^2-b^2}{b^2}\left[-\frac{b^2}{3}\left(1-\frac{y^2}{b^2}\right)^{3/2}+\frac{b^2}{3}\left(1-\frac{1}{a^2}-\frac{y^2}{b^2}\right)^{3/2}\right]_0^1$$

$$= \frac{a^2-b^2}{3}\left[1-\left(1-\frac{1}{b^2}\right)^{3/2}+\left(1-\frac{1}{a^2}-\frac{1}{b^2}\right)^{3/2}-\left(1-\frac{1}{a^2}\right)^{3/2}\right] //.$$

7. The surface is that of a cone with apex at $(0,0,1)$.



Taking V to be the cone above the (x,y) plane we have by the divergence theorem

$$\iiint_V \nabla\cdot \underline{F}\, dV = \iint_S \underline{F}\cdot\underline{n}\, dS + \iint_{S_{base}} \underline{F}\cdot\underline{n}\, dS' \quad (*)$$

$$\nabla\cdot\underline{F} = 3x^2+3y^2+6z \quad \Rightarrow$$

$$\iiint_V \nabla\cdot\underline{F}\, dV = \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=0}^{1-r^2}\left[3\left(r^2\cos^2\theta + r^2\sin^2\theta\right)+6z\right] r\, dr\, d\theta\, dz$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=0}^{1-r^2}\left(3r^2+6z\right) dz\, dr\, d\theta = 2\pi\int_{r=0}^{1}\left[3r^2z+3z^2\right]_0^{1-r^2} dr$$

$$= 2\pi\int_0^1 3(1-r^2)\, dr = 4\pi$$

Need also $\iint_{S_{base}} \underline{F} \cdot \underline{n} \, dS$; here $\underline{n} = (0,0,-1)$

$\Rightarrow \underline{F} \cdot \underline{n} = -(x^2+y^2)$ [note $z=0$]

$$\Rightarrow \iint_{S_{base}} \underline{F} \cdot \underline{n} \, dS = \iint -(x^2+y^2) \, dx \, dy$$

$$= - \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r^2 \, r \, dr \, d\theta = -\pi/2 .$$

From (*) we have

$$\iint_S \underline{F} \cdot \underline{n} \, dS = 4\pi + \frac{\pi}{2} = 9\pi/2 .$$

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8 Start from $\int_V \nabla \cdot \underline{F} \, dV = \int_S \underline{F} \cdot \underline{n} \, dS$ and put $\underline{F} = \underline{k} \times \underline{A}$

$$\nabla \cdot (\underline{k} \times \underline{A}) = \underline{A} \cdot \underbrace{(\nabla \times \underline{k})}_{0} - \underline{k} \cdot (\nabla \times \underline{A}) = -\underline{k} \cdot (\nabla \times \underline{A})$$

$$\underline{F} \cdot \underline{n} = (\underline{k} \times \underline{A}) \cdot \underline{n} = \underline{k} \cdot (\underline{A} \times \underline{n})$$ [Easily shown using tensor notation!]

$\Rightarrow$ we have

$$\int_V -\underline{k} \cdot (\nabla \times \underline{A}) \, dV = \int_S \underline{k} \cdot (\underline{A} \times \underline{n}) \, dS$$

$$\underline{k} \cdot \left(\int_V \nabla \times \underline{A} \, dV \right) = \underline{k} \cdot \int_S -(\underline{A} \times \underline{n}) \, dS$$

$\underline{k}$ arbitrary $\Rightarrow$ result follows.

If $A = (z, 0, 0)$, $\nabla \times \underline{A} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ z & 0 & 0 \end{vmatrix} = \underline{j}$

$$\Rightarrow \int_V \nabla \times \underline{A} \, dV = \underline{j} \, \frac{4}{3} \pi R^3$$

On the surface $|\underline{x}| = R$, $\underline{n} = \dfrac{(2x, 2y, 2z)}{(4x^2+4y^2+4z^2)^{1/2}} = \dfrac{(x, y, z)}{R}$

$$-\underline{A} \times \underline{n} = -\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ z & 0 & 0 \\ \frac{x}{R} & \frac{y}{R} & \frac{z}{R} \end{vmatrix} = \frac{z^2}{R} \underline{j} - \frac{yz}{R} \underline{k}$$

$$\int_S -\underline{A} \times \underline{n} \, dS = \int_S \left(\frac{z^2}{R} \underline{j} - \frac{yz}{R} \underline{k} \right) dS$$

Use spherical polars.

$$x = R \sin\theta \cos\varphi \quad y = R \sin\theta \sin\varphi \quad z = R\cos\theta$$

$$dS = R^2 \sin\theta \, d\theta \, d\varphi$$

$$\Rightarrow \int_S -\underline{A} \times \underline{n} \, dS = \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} \left[\frac{R^2 \cos^2\theta}{R} \underline{j} - \frac{R^2 \sin\theta \cos\theta \sin\varphi}{R} \underline{k} \right] R^2 \sin\theta \, d\theta \, d\varphi$$

$$= \underline{j} \, 2\pi R^3 \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta = \frac{4}{3} \pi R^3 \underline{j}$$

as required.

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(9) (i) $\hat{\underline{x}} = \dfrac{\rho_0}{\frac{2}{3}\pi R^3 \rho_0} \displaystyle\int_V (x, y, z) \, dV$ Spherical polars

$$= \frac{3}{2\pi R^3} \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \int_{r=0}^{R} \left(r\sin\theta\cos\varphi \, \underline{i} + r\sin\theta\sin\varphi \, \underline{j} + r\cos\theta \, \underline{k} \right) \times \Big(\overbrace{r^2 \sin\theta \, dr \, d\theta \, d\varphi}^{dV} \Big)$$

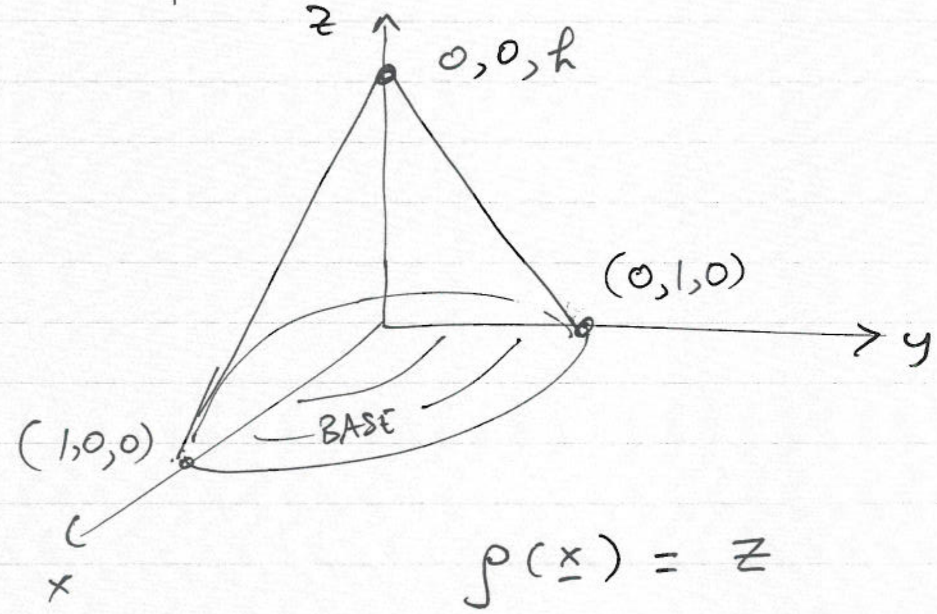
$$= \frac{3}{2\pi R^3} \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \int_{r=0}^{R} \underline{k} \left(r^3 \sin\theta \cos\theta \right) dr\, d\theta\, d\varphi$$

The $\underline{i}$, $\underline{j}$ terms are zero c.f. φ-integration 1st.

$$= \frac{3}{2\pi R^3} \left[\frac{R^4}{4} \cdot 2\pi \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta \right]$$

$$= \frac{3\underline{k}}{2\pi R^3} \cdot \frac{R^4 \pi}{2} \left[-\frac{1}{4} \cos 2\theta \right]_0^{\pi/2} = \frac{3}{8} R\, \underline{k}$$

Note: Due to symmetry we could have anticipated that the centre of mass has to lie on the z-axis!



$$\rho(\underline{x}) = z$$

$$\hat{\underline{X}} = \frac{1}{M} \int_V (x, y, z)\, z \, dV$$

Use cylindrical coords

The cone surface is given by $\frac{z}{h} = 1 - (x^2 + y^2)$

$$M = \int_V z \, dV = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=0}^{h(1-r^2)} z\, r\, dr\, d\theta\, dz = \frac{1}{6} \pi h^2$$

$$\hat{\underline{X}} = \frac{1}{M} \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=0}^{h(1-r^2)} (r\cos\theta, r\sin\theta, z)\, z\, r\, dr\, d\theta\, dz$$

$\underline{i}$, $\underline{j}$ terms zero due to θ-integration [by symmetry we can anticipate that the centre of mass is on the z-axis since $\rho(\underline{x})$ is also axisymmetric].

$$\hat{\underline{x}} = \frac{1}{M} 2\pi \int_0^1 \frac{h^3}{3}(1-r^2)^3 r\,dr = \frac{2\pi h^3}{3M}\left[-(1-r^2)^4 \frac{1}{8}\right]_0^1$$

$$= \frac{\pi h^3}{12M} = h/2 .$$

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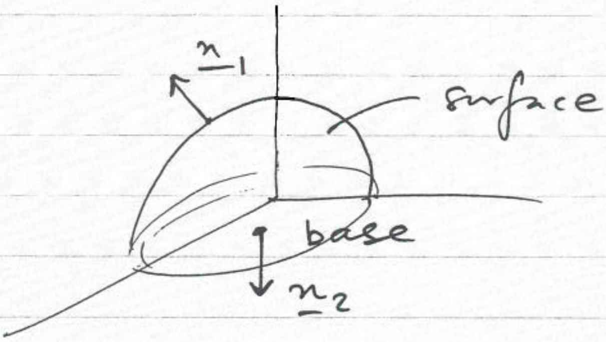

surface $r=a$ $\quad \underline{n}_1 = \frac{(x,y,z)}{a}$

base, $\underline{n}_2 = (0,0,-1)$

On the surface S, $\underline{n} = \frac{(x,y,z)}{a}$, $x^2+y^2+z^2=a^2$

$$\Rightarrow \int_S \underline{u}\cdot\underline{n}\,dS = \int_S \frac{z(z+a)}{a}\,dS$$

$$= \int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi/2} \frac{a\cos\theta\,(a\cos\theta+a)}{a}\, a^2\sin\theta\,d\theta\,d\varphi$$

$$= 2\pi a^3 \int_0^{\pi/2}\left[\cos^2\theta\sin\theta + \frac{\sin 2\theta}{2}\right]d\theta$$

$$= 2\pi a^3\left[-\frac{\cos^3\theta}{3} - \frac{\cos 2\theta}{4}\right]_0^{\pi/2} = 2\pi a^3\left(\frac{1}{3}+\frac{1}{2}\right) = \frac{5}{3}\pi a^3$$

$$\int_{S_{base}} \underline{u}\cdot\underline{n}\,dS = \int_{S_{base}} -(z+a)\,dS\Big|_{z=0} = -\int a\,dS = -\pi a^3$$

Now $\int_V \nabla \cdot \underline{u} \, dV = \int_V dV = \frac{1}{2}\left(\frac{4}{3}\pi a^3\right) = \frac{2}{3}\pi a^3$

This $= \int_S + \int_{S_{base}} (\underline{u} \cdot \underline{n}) \, dS$

$= \frac{5}{3}\pi a^3 - \pi a^3$, Theorem verified