M2AA2 - Multivariable Calculus. Assessed Coursework II
Solutions
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1. (a) Use a source singularity at $\boldsymbol{x}_0 = (x_0, y_0)$ and an image sink singularity at $\boldsymbol{x}_0' = (x_0, -y_0)$ to find the required Green's function:

$$G(\boldsymbol{x};\boldsymbol{x}_0) = \frac{1}{2\pi}\log|\boldsymbol{x}-\boldsymbol{x}_0| - \frac{1}{2\pi}\log|\boldsymbol{x}-\boldsymbol{x}_0'|.$$

(b) The problem is a Dirichlet one, therefore the solution in terms of the Dirichlet Green's function at any point $\boldsymbol{x}_0$ in the upper half-plane is

$$\phi(\boldsymbol{x}_0) = \int_{\partial D}\phi\frac{\partial G}{\partial n}ds = \int_{-\infty}^{\infty}\phi(x)\left[-\frac{\partial G}{\partial y}\right]dx. \qquad (1)$$

Now substitute $\left[-\frac{\partial G}{\partial y}\right]_{y=0} = \frac{(y_0/\pi)}{(x-x_0)^2+y_0^2}$ and the boundary conditions into the solution (1) to obtain

$$\phi(x_0,y_0) = \frac{y_0}{\pi}\int_{-1}^{1}\frac{dx}{(x-x_0)^2+y_0^2}. \qquad (2)$$

(c) The integral in (2) can be carried out in closed form

$$\phi(x_0,y_0) = \left[\frac{1}{\pi}\tan^{-1}\left(\frac{x-x_0}{y_0}\right)\right]_{-1}^{1} = \frac{1}{\pi}\left[\tan^{-1}\left(\frac{1-x_0}{y_0}\right)+\tan^{-1}\left(\frac{1+x_0}{y_0}\right)\right]. \qquad (3)$$

- $x_0 > 1$: Consider the limit $y_0 \to 0+$ in (3). The first term tends to $-\pi/2$ while the second tends to $\pi/2$, hence the solution tends to 0 as it should according to the boundary conditions.
- $x_0 < -1$: As above, but now the first term tends to $+\pi/2$ while the second tends to $-\pi/2$.
- $-1 < x_0 < 1$: Both terms tend to $+\pi/2$ as $y_0 \to 0+$, hence $\phi \to 1$.

(d) First part follows immediately from setting $x_0 = 0$ in (3).
A reduction by a factor N implies

$$\frac{1}{N} = \frac{2}{\pi}\tan^{-1}\left(\frac{1}{y_N}\right) \quad \Rightarrow \quad y_N = \frac{1}{\tan(\pi/2N)}, \qquad (4)$$

as required.
A reduction by 99% means that the temperature will be 1/100 of what it was to begin with at the wall. Hence $N = 100$. Since $\tan\epsilon = \epsilon + \ldots$ and $\pi/200 \ll 1$, we estimate $y_{100} \approx 200/\pi \approx 70$.

2. (a) Write $u(x,t) = X(x)T(t)$ which on separation of variables gives

$$\frac{1}{\kappa}\frac{T'(t)}{T} = \frac{X''(x)}{X}.$$

For decaying solutions at large times, the separation constant must be negative, i.e.

$$T' = -\kappa\lambda^2, \quad X'' + \lambda^2 X = 0.$$

The solution for X is

$$X(x) = \alpha\sin(\lambda x) + \beta\cos(\lambda x),$$

and since $X(0) = X(L) = 0$ from the boundary conditions, we have $\beta = 0$ and $\lambda = n\pi/L$. Hence

$$X_n(x) = \alpha_n \sin(n\pi x/L),$$

is a separated solution for any $n \geq 1$.

With this value of λ we can solve for T to find

$$T_n(t) = A_n \exp(-n^2\pi^2\kappa t/L^2),$$

as the time-dependent separated solution.

Putting these together gives the required series solution.

(b) To find s_n we impose the initial condition to obtain

$$U_0 = \sum_{n=1}^{\infty} s_n \sin\frac{n\pi x}{L}.$$

This is the fourier sine series of U_0 and hence we have

$$s_n = \frac{2}{L}\int_0^L U_0 \sin\frac{n\pi x}{L} dx = \frac{2U_0}{n\pi}(1-\cos n\pi).$$

If $\kappa = L = 1$, then $u(x,t) = s_1 \sin(\pi x)\exp(-\pi^2 t) + \ldots = \frac{4U_0}{\pi}\sin(\pi x)\exp(-\pi^2 t) + \ldots$. Hence, the required time is approximately given by

$$\frac{4U_0}{\pi}\exp(-\pi^2 t) = \frac{U_0}{2} \quad \Rightarrow \quad t \approx \frac{1}{\pi^2}\ln(8/\pi).$$

3. The argument is correct until the step involving equation (4). Here are the reasons:

- The cosine series of e^x is valid for $x \in [0, L]$ and comes from the fourier series of e^x on $[-L, L]$ when e^x for $x \in [0, L]$ is extended to $[-L, 0]$ as an *even* function. Hence, the function is piecewise smooth and continuous.

- The expression (3) can therefore be differentiated term-by-term and will converge everywhere to e^x except at the points where the derivative is discontinuous, i.e. $x = 0, L$. This is the fourier sine series of e^x - we can see directly from thinking about the fourier sine series of e^x, that there will be discontinuities at $x = 0, L$ since it must be extended to $[-L, 0]$ as an *odd* function of x.
- If we now take this fourier sine series, we cannot differentiate it term-by-term because it is not continuous. This is where the argument goes wrong.
- Fourier coefficients can be found in the usual way.