M2AA2 - Multivariable Calculus. Assessed Coursework I
Solutions
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1. (a) The force exerted on P by mass k is given by $m_k/r_k = m_k/\sqrt{(x-\xi_k)^2+(y-\eta_k)^2+(z-\zeta_k)^2}$. Due to linearity we add the resulting n forces and the result follows.

   (b) Units: μ_R has units of mass/volume; μ_S has units of mass/unit area; μ_C has units of mass per unit length.

   To show (2)a, take a volume element dV around the point $\boldsymbol{\xi}$ in 3-dimensional space; it has mass $\mu(\boldsymbol{\xi})dV$ and hence exerts a force $\mu(\boldsymbol{\xi})dV/|\boldsymbol{x}-\boldsymbol{\xi}|$ on a given point P. Now we need to add all these forces up which means taking an integral over the volume where the mass density has its support. The result follows.

   To show (2)b-c we proceed analogously now taking integrals over the given surface S or given curve C where μ has its support. Note that the position of P is anywhere in space, not necessarily inside V, on S or on C for each of the three cases, and so V is a function of $\boldsymbol{x} \in \boldsymbol{R}^3$.

   (c) The required potential is given by

$$
\begin{aligned}
V &= \int_{sphere} \frac{d\xi\, d\eta\, d\zeta}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}} \\
&= \int_{-1}^{1} d\xi \int_{-\sqrt{1-\xi^2}}^{\sqrt{1-\xi^2}} d\eta \int_{-\sqrt{1-\xi^2-\eta^2}}^{\sqrt{1-\xi^2-\eta^2}} \frac{d\zeta}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}
\end{aligned}
$$

   The last integral can be carried out; it has the form $\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\frac{u}{a}$, where we can identify $a^2 = (x-\xi)^2+(y-\eta)^2$ and $u = z-\zeta$ by substitution. This is as far as analytical progress goes.

   (d) Begin by considering the point P to be outside of region R so that $r \neq 0$. Then, we can differentiate under the integral sign and use the fact that $\boldsymbol{F} = \nabla V_R$, etc.. This gives

$$
\boldsymbol{F} = -\int\int\int_R \frac{(x-\xi,\, y-\eta,\, z-\zeta)}{r^3} d\xi\, d\eta\, d\zeta, \quad r = \sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}.
$$

   If the point P is outside of R then the integrals are well-defined.

   If P is inside the region then the integrand is singular at the point $\xi = x$, $\eta = y$, $\zeta = z$. To analyse whether the integral has a limit we use spherical polar coordinates centred at (x, y, z), i.e. write

$$
\xi - x = \epsilon \sin\theta\cos\varphi, \;\; \eta - y = \epsilon\sin\theta\sin\varphi, \;\; \zeta - z = \epsilon\cos\theta.
$$

   (I use ϵ instead of r to emphasize that we are interested in the limit $\epsilon \to 0$.) Take any one of the forces, e.g.

$$
F_1 = \int\int\int \frac{\epsilon\sin\theta\cos\phi}{\epsilon^3}\epsilon^2 d\epsilon d\theta d\varphi,
$$

   and clearly this has no problems as $\epsilon \to 0$, hence the integrals are convergent.

(e) i. The required integral is

$$V_S = \int\int_{sphere} \frac{dS}{\sqrt{(x-\xi)^2+\eta^2+\zeta^2}}.$$

Now introduce spherical polars to parametrise the surface of the sphere:

$$\begin{aligned}\xi &= a\cos\theta,\\ \eta &= a\sin\theta\cos\varphi,\\ \zeta &= a\sin\theta\sin\varphi.\end{aligned}$$

Note that this choice helps with the integration. It is equivalent to rotating the familiar polar coordinates system so that the $\theta = 0$ axis contains the point P. (You can see how this works without a rotation if the point P is at $(0, 0, z)$, for example.) Anyway, the integral becomes

$$\begin{aligned}V_S &= \int_0^{\pi} \frac{a^2\sin\theta}{\sqrt{(x-a\cos\theta)^2+a^2\sin^2\theta}}d\theta \int_0^{2\pi} d\varphi\\ &= 2\pi\int_0^{\pi} \frac{a^2\sin\theta}{\sqrt{x^2+a^2-2ax\cos\theta}}d\theta\end{aligned}$$

Now substitute $x^2 + a^2 - 2ax\cos\theta = r^2$ and as long as $x \neq 0$ we get

$$V_S = \frac{2\pi a}{x}\int_{|x-a|}^{|x+a|} dr = \frac{2\pi a}{x}(|x+a|-|x-a|).$$

ii. If P is outside the sphere, i.e. $|x| > a$, we have

$$V_S = \frac{4\pi a^2}{|x|},$$

and so the potential is as if all the mass $(4\pi a^2)$ is concentrated at the origin.
If $|x| < a$, $V_S = 4\pi a$, i.e. a constant.

iii. From the results above the potential across the surface is continuous.
Considering the x-component of the force (recall that the force is the gradient of the potential), we can calculate this to be $-4\pi a^2/x^2$ for $|x| > a$ and 0 for $|x| < a$. Hence there is a jump by an amount -4π in crossing the surface from outside to inside.

2. A graph of the region R is in the figure.

To calculate the area with the given change of variables we first need to identify the mapped region: The point $x = 0, y = 0$ maps to $u = 0, v = 0$. The point $x = 2, y = 1$ maps to $u = 1, v = 1$ and the point $x = 1, y = 0$ maps to $u = 1, v = 0$.

The line $y = x/2$ is $u^2 + v^2 - 2uv = (u - v)(u + v) = 0$. Given the position of the mapped points and that this line connected $(0, 0)$ to $(2, 1)$ in the (x, y) plane this maps to $u = v$.

The curve $x = 1 + y^2$ is $u^2 + v^2 = 1 + u^2v^2$ and this is satisfied by $u = \pm 1$ and from the mapped points $u = 1$.

The line $y = 0$ is $uv = 0$ and again from the mapped vertices this is $u = 0$.

Thus the triangle is defined by $0 \le u \le 1$ and $0 \le v \le u$.

Now for the integral we require the Jacobian

$$J = \frac{\partial(x, y)}{\partial(u, v)} = 2(u^2 - v^2)$$

and the integral I is

$$I = \int_R dxdy = \int_{Rnew} Jdudv = \int_0^1 \int_0^u 2(u^2 - v^2)dvdu = 2\int_0^1 \frac{2}{3}u^3 du = \frac{1}{3}.$$

Set $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ then the unit vectors in the u, v directions are proportional to

$$\frac{\partial \mathbf{r}}{\partial u} = 2u\mathbf{i} + v\mathbf{j}, \quad \frac{\partial \mathbf{r}}{\partial v} = 2v\mathbf{i} + u\mathbf{j}.$$

If this were an orthogonal coordinate system then the dot product of these quantities would be zero - it is n't. This is not orthogonal.

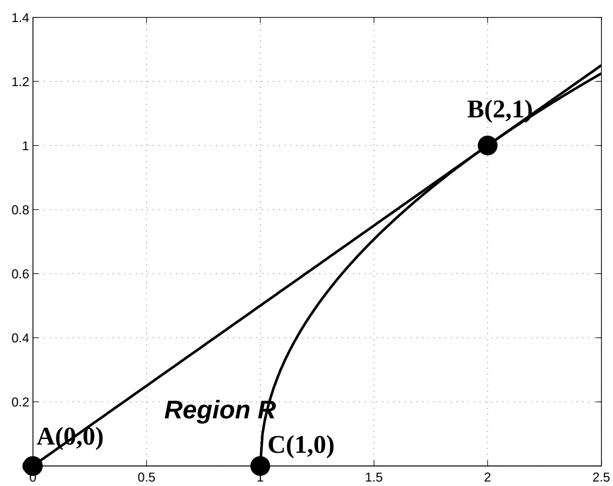

Figure 1: The region R for Problem 2

3. (a) Bookwork:

$$\int\int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n} dS = \oint_C \mathbf{A} \cdot d\mathbf{r}$$

where C is the boundary of S orientated in a positive sense.

Unseen: The integral is

$$\oint_C \mathbf{A} \cdot d\mathbf{r}$$

where C is the circular path in the xy plane, origin at (0,0) and radius 3 in (i) and 1 in (ii). So do both simultaneously

$$x = R\cos\theta, \quad y = R\sin\theta$$

for $0 \le \theta \le 2\pi$. The integral is then

$$\int_0^{2\pi} -[4R^2\cos^2\theta + R\sin\theta - 3]R\sin\theta + R\cos\theta[5R^2\cos\theta\sin\theta]d\theta = -\int_0^{2\pi} R^2\sin^2\theta d\theta = -R^2\pi$$

so result for (i) is -9π and for (ii) is $-\pi$.

(b) Bookwork

$$\int\int_S \mathbf{A}\cdot\mathbf{n}dS = \int\int\int_V \nabla\cdot\mathbf{A}dV$$

where S is the surface enclosing the volume V.

Unseen: The integral is

$$\int\int\int_V \nabla\cdot\mathbf{A}dV = \int\int\int_V (8+2y)dV = 1\times 1\times\int_0^1 (8+2y)dy = 9$$