M2AA2 - Multivariable Calculus. Problem Sheet 8. Solutions.
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Recall that the Euler equation that extremises $I[y] = \int_{x_0}^{x_1} f(x, y, y')dx$ is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0. \tag{1}$$

If the function $f$ is independent of $x$, then the following is an integral of the Euler equation above,

$$f(y, y') - y'\frac{\partial f}{\partial y'}(y, y') = const. \tag{2}$$

These results are used in the problems that follow.

1. (a) The integrand is $f(x, y, y') = (y')^2/s^3$. The Euler equation (1) is, therefore,

$$-\frac{d}{dx}\left(\frac{2y'}{x^3}\right) \quad \Rightarrow \quad y(x) = K_1x^4 + K_2.$$

Apply boundary conditions to find $y(x) = (1/5)(-x + 1)$.

(b) Since $f$ is independent of $x$ we can use (2), which becomes

$$-\frac{1}{2}(y')^2 + y = c \quad \Rightarrow \quad \frac{dy}{\sqrt{y-c}} = \sqrt{2}dx \quad \Rightarrow$$
$$\sqrt{2}(y-c)^{1/2} = x + d \quad \Rightarrow \quad 2(y-c) = (x+d)^2.$$

Apply boundary conditions to find $d = 1/2$, $c = -1/8$.

2. The Euler equation (1) and its solution is

$$-\frac{d}{dx}\left[\frac{y'}{x\sqrt{1+(y')^2}}\right] = 0 \quad \Rightarrow \quad \frac{y'}{x\sqrt{1+(y')^2}} = c$$
$$\Rightarrow \quad (y')^2 = \frac{x^2c^2}{1-x^2c^2} \quad \Rightarrow \quad y + d = -\frac{1}{c}\sqrt{1-x^2c^2} \quad \Rightarrow \quad x^2 + (y+d)^2 = \frac{1}{c^2}.$$

Now use the boundary conditions to find $c, d$; we find $d = \frac{x_0^2-x_1^2+y_0^2-y_1^2}{2(y_1-y_0)}$ with $c$ following from either of the equations $x_0^2 + (y_0+d)^2 = 1/c^2$ OR $x_1^2 + (y_1+d)^2 = 1/c^2$.

3. Let us do part (b) and then part (a) is a special case. Include variations to an optimal solution $u(x)$ by writing $y(x) = u(x) + \epsilon\eta(x)$. The condition for an extremum is (see your notes)

$$0 = I'(\epsilon) = \int_{x_0}^{x_1}\left[\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta' + \dots \frac{\partial f}{\partial y^{(n)}}\eta^{(n)}\right]dx$$

Now integrate by parts each term to write the integral as $\int_{x_0}^{x_1}\eta(x)L[y]dx$, hence $L[y] = 0$ as required. In the integration by parts, all derivatives up to and including $\eta^{(n-1)}$ vanish at $x = x_0, x_1$. The only thing you need to convince yourselves with is the result

$$\int_{x_0}^{x_1}\eta^{(n)}\frac{\partial f}{\partial y^{(n)}}dx = \eta^{(n-1)}\frac{\partial f}{\partial y^{(n)}}\Big|_{x_0}^{x_1} - \int_{x_0}^{x_1}\eta^{(n-1)}\frac{d}{dx}\frac{\partial f}{\partial y^{(n)}}dx$$
$$= -\,\eta^{(n-2)}\frac{d}{dx}\frac{\partial f}{\partial y^{(n)}}\Big|_{x_0}^{x_1} + \int_{x_0}^{x_1}\eta^{(n-2)}\frac{d^2}{dx^2}\frac{\partial f}{\partial y^{(n)}}dx = \dots = \int_{x_0}^{x_1}(-1)^n\frac{d^n}{dx^n}\frac{\partial f}{\partial y^{(n)}}.$$

4. Here $f$ is a function of $x, y, y'$ so use the Euler equation (1). This becomes

$$2by' + 2cy - \frac{d}{dx}\left[2ay' + 2by\right] = 0 \quad \Rightarrow \quad ay'' + a'y' + (b' - c)y = 0,$$

is the desired 2nd order ODE. If $b = const.$ then this does not enter into the equation since $b' = 0$.

5. The Euler equation comes from Problem 3 with $n = 2$. It is in this case

$$32y - \frac{d}{dx}\left[0\right] + \frac{d^2}{dx^2}\left[2y''\right] = 0 \quad \Rightarrow \quad y'''' + 16y = 0.$$

Looking for solutions of the form $\exp(\lambda x)$ gives $\lambda^4 = 16$, i.e. $\lambda = \pm 2, \pm 2i$, hence

$$y(x) = Ae^{2x} + Be^{-2x} + C\cos 2x + D\sin 2x.$$

Now apply the boundary conditions to get $A, B, C, D$.

6. In this problem we are dealing with stationary values of integrals of $F \equiv F(x, \phi, \phi_x, \phi_y, \phi_{xy}, \phi_{xx}, \phi_{yy})$. There are two independent variables $x, y$ but $F$ depends on derivatives of $\phi$ up to and including 2nd order. In your notes we derived the Euler equations for $F(x, \phi, \phi_x, \phi_y)$; this problem extends that.

Including variations to the stationary function via $\phi(x, y) = u(x, y) + \epsilon\eta(x, y)$ leads to the integral condition ($I'(\epsilon) = 0$ in the notation of the notes)

$$\int\int_D \left[\eta\frac{\partial F}{\partial \phi} + \eta_x\frac{\partial F}{\partial \phi_x} + \eta_y\frac{\partial F}{\partial \phi_y} + \eta_{xy}\frac{\partial F}{\partial \phi_{xy}} + \eta_{xx}\frac{\partial F}{\partial \phi_{xx}} + \eta_{yy}\frac{\partial F}{\partial \phi_{yy}}\right] dxdy.$$

We need to integrate by parts to cast the integral above into the form $\int\int_D \eta(x, y)L[\phi]dxdy$. The way to deal with the first three terms is in your notes. I will show you how to integrate the 4th term, the 5th and 6th being essentially the same. Start with defining the upper and lower boundaries of $D$ as $y = U_2(x)$ and $y = U_1(x)$, respectively, and also the right and left boundaries by $x = R_2(y)$ and $x = R_1(y)$. Then

$$\int\int_D \eta_{xy}\frac{\partial F}{\partial \phi_{xy}} dxdy = \int_x \left[\eta_x \frac{\partial F}{\partial \phi_{xy}}\right]_{U_1(x)}^{U_2(x)} dx - \int\int_D \eta_x \frac{\partial}{\partial y}\frac{\partial F}{\partial \phi_{xy}} dxdy$$

The boundary terms are zero, hence the first integral on the RHS is zero. Continue with an integration by parts with respect to $x$ now.

$$-\int\int_D \eta_x \frac{\partial}{\partial y}\frac{\partial F}{\partial \phi_{xy}} dxdy = -\int_y \left[\eta \frac{\partial^2 F}{\partial y \partial \phi_{xy}}\right]_{R_1(y)}^{R_2(y)} dy + \int\int_D \eta \frac{\partial^3 F}{\partial x \partial y \partial \phi_{xy}} dxdy.$$

Again the boundary term is zero leaving the result

$$\int\int_D \eta_{xy}\frac{\partial F}{\partial \phi_{xy}} dxdy = \int\int_D \eta \frac{\partial^3 F}{\partial x \partial y \partial \phi_{xy}} dxdy.$$

We can get the other two integrals in a totally analogous way - note that since there is an even number of derivatives of $\eta$ in each of these terms the sign of the required integral is positive. [Carry out the work for the other two terms to familiarise yourselves with the mechanics of it.] Put it all together to obtain

$$\int\int_D \eta(x,y)\left[\frac{\partial F}{\partial\phi}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial\phi_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial\phi_y}\right)+\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial F}{\partial\phi_{xy}}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial\phi_{xx}}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial\phi_{yy}}\right)\right]dxdy,$$

leading to the following Euler equation (the usual arguments/Lemmas are used here)

$$\frac{\partial F}{\partial\phi}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial\phi_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial\phi_y}\right)+\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial F}{\partial\phi_{xy}}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial\phi_{xx}}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial\phi_{yy}}\right)=0. \qquad (3)$$

Now we are in a position to answer the problems:

(a) Here $F=\phi_{xx}^2+\phi_{yy}^2+2\phi_{xx}\phi_{yy}$ and hence depends only on $\phi_{xx}$ and $\phi_{yy}$. The Euler equation (3) becomes, therefore,

$$\frac{\partial^2}{\partial x^2}(2\phi_{xx}+2\phi_{yy})+\frac{\partial^2}{\partial y^2}(2\phi_{yy}+2\phi_{xx})=0 \quad\Rightarrow\quad \Delta^2\phi=0, \quad \text{where} \quad \Delta\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.$$

The operator $\Delta^2=\nabla^4$ is called the *biharmonic* operator.

(b) In this case $F=\phi_{xx}^2+\phi_{yy}^2+3\phi_{xx}\phi_{yy}-\phi_{xy}^2$, and the Euler equation (3) becomes

$$(-2\phi_{xy})_{xy}+(2\phi_{xx}+3\phi_{yy})_{xx}+(2\phi_{yy}+3\phi_{xx})_{yy}=0 \quad\Rightarrow\quad \phi_{xxxx}+2\phi_{xxyy}+\phi_{yyyy}=0$$
$$\Rightarrow\quad \Delta^2\phi=0$$

7. We introduce a Lagrange multiplier $\lambda$ and note that $a,b,c$ are functions of $x$, hence the Euler equation is

$$2bu'+2cu+2\lambda u-\frac{d}{dx}(2au'+2bu)=0 \quad\Rightarrow\quad au''+a'u'+(b'-c-\lambda)u=0,$$

is the required Euler equation.

8. First consider the general case of finding the geodesics of a given surface $G(x,y,z)=0$, say. Parametrise the curve by $x=x(t)$, $y=y(t)$, $z=z(t)$ and the problem is that of minimizing

$$\int_{t_0}^{t_1}F(x,y,z,\dot{x},\dot{y},\dot{z})dt, \qquad F(x,y,z,\dot{x},\dot{y},\dot{z})=\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2} \qquad (4)$$

subject to

$$G(x(t),y(t),z(t))=0. \qquad (5)$$

In addition to (5) the required conditions are

$$\frac{d}{dt}\frac{\partial F}{\partial\dot{x}}-\frac{\partial F}{\partial x}=\lambda\frac{\partial G}{\partial x}$$
$$\frac{d}{dt}\frac{\partial F}{\partial\dot{y}}-\frac{\partial F}{\partial y}=\lambda\frac{\partial G}{\partial y}$$
$$\frac{d}{dt}\frac{\partial F}{\partial\dot{z}}-\frac{\partial F}{\partial z}=\lambda\frac{\partial G}{\partial z}$$

which on substituting $F$ from (4) become

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}} = \lambda\frac{\partial G}{\partial x} \tag{6}$$

$$\frac{d}{dt}\frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}} = \lambda\frac{\partial G}{\partial y} \tag{7}$$

$$\frac{d}{dt}\frac{\dot{z}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}} = \lambda\frac{\partial G}{\partial z} \tag{8}$$

Now for a cylinder (without loss of generality take the radius to be 1), we have

$$G(x,y,z) = x^2+y^2-1 = 0, \tag{9}$$

i.e. $G$ does not depend on $z$. This implies that (8) becomes

$$\frac{d}{dt}\frac{\dot{z}}{\sqrt{1+\dot{z}^2}} = 0 \qquad \Rightarrow \qquad \dot{z} = const.$$

Since the cylinder is parametrised by $x = x(t) = \cos t$, $y = y(t) = \sin t$ (follow from (9)), and $z = z(t) = ct$ with $c$ a constant, the solution is a helix. Equations (6) or (7) can be used to find $\lambda$.

9. Suppose that the required closed curve has parametric equations $x = x(t)$ and $y = y(t)$, $t_0 \le t \le t_1$. From Green's theorem the enclosed area is given by

$$I = \frac{1}{2}\int_{t_0}^{t_1}(x\dot{y} - y\dot{x})dt. \tag{10}$$

The constraint is that the curve has given length, i.e.

$$\int_{t_0}^{t_1}\sqrt{\dot{x}^2+\dot{y}^2}dt = L.$$

Note that the dependent variables are now $x, \dot{x}, y\dot{y}$ and the independent variable is $t$. Hence, the Euler equations introducing a Lagrange multiplier are

$$\frac{1}{2}\dot{y} - \frac{d}{dt}\left[-\frac{1}{2}y + \frac{\lambda\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right] = 0$$

$$\frac{1}{2}\dot{x} + \frac{d}{dt}\left[\frac{1}{2}x + \frac{\lambda\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right] = 0$$

Now use arc length to parametrise the curve, i.e. set $t = s$ above. For an arc length parametrisation we have

$$x'^2 + y'^2 = 1,$$

where primes denote $d/ds$, hence the differential equations become

$$\begin{aligned} y' - \lambda x'' &= 0 \\ x' + \lambda y'' &= 0 \end{aligned}$$

Integrate these to obtain the solutions

$$\begin{aligned} \lambda x' &= y + K_1 \\ \lambda y' &= K_2 - x \end{aligned}$$

and since $x'^2 + y'^2 = 1$ we have

$$(x-\alpha)^2 + (y-\beta)^2 = \lambda^2,$$

which is the equation for a circle. The constants $\alpha$, $\beta$, $\lambda$ can be determined from the supplied conditions.