M2AA2 - Multivariable Calculus. Problem Sheet 3. Solutions.
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1. (a) The centroid of a region S is given by $\overline{x} = \frac{1}{A}\int\int_S x dxdy$, $\overline{y} = \frac{1}{A}\int\int_S y dxdy$, where A is the area of the region. Now use Green's theorem in the plane, $\int\int_S (F_{2x} - F_{1y})dxdy = \oint_C (F_1 dx + F_2 dy)$ with $F_1 = 0, F_2 = \frac{x^2}{2}$ and $F_1 = \frac{y^2}{2}, F_2 = 0$, respectively, to get the required result.

   (b) For S_1, parametrise using $x = a\cos\theta$, $y = a\sin\theta$, to obtain $\overline{x} = \frac{1}{\pi a^2}\int_0^{2\pi} a^2 \sin^2\theta(\cos\theta)d\theta = 0$ with an analogous result for $\overline{y}$.

   For S_2 we now have two parts contributing to C since the region is not simply connected (see class notes on how we prove Green's theorem for regions which are not simply connected). Can parametrise on either one by $(x, y) = r(\cos\theta, \sin\theta)$ with $r = a$ or b. Then each integral is zero as found earlier.

2. Calculate $\nabla\cdot(\nabla\phi\times\nabla\psi) = \nabla\psi\cdot(\nabla\times\nabla\phi) - \nabla\phi\cdot(\nabla\times\nabla\psi)$ and since $\nabla\times(\nabla\phi) = 0 = \nabla\times(\nabla\psi)$, the result follows.

   Take $\nabla\times\frac{1}{2}(\phi\nabla\psi - \psi\nabla\phi) = \frac{1}{2}(\nabla\phi\times\nabla\psi - \nabla\psi\times\nabla\phi) = \nabla\phi\times\nabla\psi$ which is what we need to show.

3. The divergence theorem with $\boldsymbol{F} = \boldsymbol{u}\times\boldsymbol{K}$ is $\int_V \nabla\cdot(\boldsymbol{u}\times\boldsymbol{K})dV = \int_S (\boldsymbol{u}\times\boldsymbol{K})\cdot\boldsymbol{n}dS$. Now use the identity

$$\nabla\cdot(\boldsymbol{u}\times\boldsymbol{K}) = \boldsymbol{K}\cdot(\nabla\times\boldsymbol{u}) - \boldsymbol{u}\cdot(\nabla\times\boldsymbol{K}) = \boldsymbol{K}\cdot(\nabla\times\boldsymbol{u}).$$

   Next use $(\boldsymbol{u}\times\boldsymbol{K})\cdot\boldsymbol{n} = \boldsymbol{K}\cdot(\boldsymbol{n}\times\boldsymbol{u})$, to write the divergence theorem form above as

$$\boldsymbol{K}\cdot\int_V (\nabla\times\boldsymbol{u})dV = \boldsymbol{K}\cdot\int_S \boldsymbol{n}\times\boldsymbol{u}dS,$$

   and since $\boldsymbol{K}$ is arbitrary the result follows.

4. The divergence theorem for $\phi\boldsymbol{F}$ is

$$\int_V \nabla\cdot(\phi\boldsymbol{F})dV = \int_S \phi\boldsymbol{F}\cdot\boldsymbol{n}dS$$

   i.e.

$$\int_V (\nabla\phi\cdot\boldsymbol{F} + \phi\nabla\cdot\boldsymbol{F})dV = \phi_0\int_S \boldsymbol{F}\cdot\boldsymbol{n}dS.$$

   Now since $\nabla\cdot\boldsymbol{F} = 0$ we have $\int_V \nabla\cdot\boldsymbol{F}dV = \int_S \boldsymbol{F}\cdot\boldsymbol{n} = 0$ which when substituted above give the answer.

5.
$$\int_S (\boldsymbol{r}\cdot\boldsymbol{n})dS = \int_V \nabla\cdot\boldsymbol{r}dV = 3\int_V dV = 3V,$$

   where V is the volume enclosed by S.

6. $\nabla\cdot\boldsymbol{F} = 1$ hence in the divergence theorem $\int_V \nabla\cdot\boldsymbol{F}dV = (2a)^3$, i.e. the volume of the cube. Need to consider $\int_S \boldsymbol{F}\cdot\boldsymbol{n}dS$. Normals point *out* of the volume, hence $\boldsymbol{F}\cdot\boldsymbol{n} = a$ for the faces $x = \pm a$ and $\boldsymbol{F}\cdot\boldsymbol{n} = 0$ on all other faces. Over the face at $x = \pm a$ we have $\int_S a dS = a(2a)^2$, so $\int \boldsymbol{F}\cdot\boldsymbol{n}dS = 2a(2a)^3$ as required.

7. Need to find $\int_S \boldsymbol{F}\cdot\boldsymbol{n}dS$.

   (a) If O lies outside S then we have

$$\int_S \left(-GM\frac{\boldsymbol{r}\cdot\boldsymbol{n}}{r^3}dS\right) = -GM\int_V \nabla\cdot\left(\frac{\boldsymbol{r}}{r^3}\right)dV = 0,$$

   since $\nabla\cdot\left(\frac{\boldsymbol{r}}{r^3}\right) = 0$ if $\boldsymbol{r}\neq 0$ as is the case here.

   (b) If the origin is inside S, then we surround it by a small sphere of radius ϵ and call the surface of this sphere S_ϵ. Then

$$\int_{S+S_\epsilon} \boldsymbol{F}\cdot\boldsymbol{n} = 0,$$

   since the result (a) holds - in the volume $V - V_\epsilon$ we have $\nabla\cdot\left(\frac{\boldsymbol{r}}{r^3}\right) = 0$.

   Hence (note that $\boldsymbol{n} = -\hat{\boldsymbol{r}}$)

$$\int_S \boldsymbol{F}\cdot\boldsymbol{n}dS = -\int_{S_\epsilon}\boldsymbol{F}\cdot\boldsymbol{n}dS = -GM\int_{S_\epsilon}\frac{r\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{r}}}{r^3}dS = -4\pi GM.$$

   This tells us that

$$\nabla\cdot\boldsymbol{F} = -\frac{3GM}{a^3}\delta(\boldsymbol{r}).$$

8. We are in spherical polar coordinates and want to evaluate integrals. Hence, $dS = a^2\sin\theta d\theta d\varphi$, and

$$\begin{aligned} I_1 &= \int_0^{2\pi}\int_0^{\pi} a^4\sin^3\theta\cos^2\varphi d\theta d\varphi = a^4\int_0^{2\pi}\cos^2\varphi\int_0^{\pi}(1-\cos^2\theta)\sin\theta d\theta \\ &= a^4\int_0^{2\pi}\cos^2\varphi\left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^{\pi}d\varphi = \frac{4}{3}a^4\int_0^{2\pi}\cos^2\varphi = \frac{4}{3}\pi a^4. \end{aligned}$$

   I_3 is similar and gives

$$I_3 = \int_0^{2\pi}\int_0^{\pi} a^4\cos^2\theta\sin\theta d\theta d\varphi = 2\pi\left[-\frac{1}{3}a^4\cos^3\theta\right]_0^{\pi} = \frac{4}{3}\pi a^4.$$

   If we consider

$$\int_S (x_1^2 + x_2^2 + x_3^2)dS = a^2\int_S dS = 4\pi a^4.$$

   Now each of I_1, $I_2 = \int_S x_2^2 dS$ and I_3 are equal and thus $4\pi a^4/3$.