M2AA2 - Multivariable Calculus. Problem Sheet 7
March 16, 2009. Prof. D.T. Papageorgiou

1. (a) Find the Green's function

$$\nabla^2 G = \delta(\boldsymbol{x} - \boldsymbol{x}_0), \qquad x > 0,\ y > 0$$
$$G(0, y) = 0, \qquad \frac{\partial G}{\partial y}(x, 0) = 0.$$

(b) Use part (a) to solve the problem

$$\nabla^2 \phi = f(x, y), \qquad x > 0,\ y > 0$$
$$\phi(0, y) = q(y), \qquad \frac{\partial \phi}{\partial y}(x, 0) = p(x).$$

2. Use the method of images (there will be a countably infinite number of them in the problems that follow) to obtain the Green's function $G(\boldsymbol{x}; \boldsymbol{x}_0)$

$$\nabla^2 G = \delta(\boldsymbol{x} - \boldsymbol{x}_0)$$

for the following problems:

(a) On the rectangle $0 < x < L$, $0 < y < H$ if $G = 0$ at $x = 0$ and $x = L$ and $\partial G/\partial y = 0$ at $y = 0$ and $y = H$.

(b) On the infinite strip $0 < x < L$, $-\infty < y < \infty$ if $G = 0$ at $x = 0$ and $\partial G/\partial y = 0$ at $x = L$.

(c) On the infinite strip $0 < x < L$, $-\infty < y < \infty$, $-\infty < z < \infty$ if $G = 0$ at $x = 0$ and $G = 0$ at $x = L$.

(d) On the semi-infinite strip $0 < x < L$, $0 < y < \infty$ if $G = 0$ along the boundaries.

(e) On the semi-infinite strip $0 < x < L$, $-\infty < y < 0$ if $G = 0$ at $x = 0$, $G = 0$ at $x = L$, $\partial G/\partial y = 0$ at $y = 0$.

3. Consider the following boundary-value problem for Laplace's equation in a circle:

$$\begin{aligned} \nabla^2 \phi &= 0, \qquad x^2 + y^2 < R, \\ \phi(R, \theta) &= f(\theta). \end{aligned}$$

(a) Use separation of variables to find all possible solutions and hence show that the solution can be written as

$$\phi(r, \theta) = \frac{1}{\pi} \int_0^{2\pi} f(\alpha) \left[ \frac{1}{2} + \sum_{n=1}^{\infty} \frac{r^n}{R^n} \cos(\theta - \alpha) \right] d\alpha, \tag{1}$$

and justify any integration and summation interchanges that you may have performed.

(b) By summing the infinite series in (1) show that the solution is

$$\phi(r, \theta) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f(\alpha)}{R^2 - 2Rr\cos(\theta - \alpha) + r^2} d\alpha.$$

This solution is known as *Poisson's integral formula.*

(c) Show directly from your separation of variables solution (not the Poisson formula) that if $f(\theta) = 1$, then the solution is $\phi(r, \theta) = 1$. Now verify that this is also the case from the Poisson integral formula.

4. The wave equation for $u(x, t)$ in one dimension is given by (subscripts denote partial derivatives, e.g. $u_{tt} = \frac{\partial^2 u}{\partial t^2}$ etc.)

$$u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \; t > 0,$$

where $c$ is a real constant, and initial conditions at $t = 0$ need to be specified also.

(a) Show that the change of independent variables $(x, t) \to (\xi, \eta)$ where $\xi = x - ct$, $\eta = x + ct$, casts the wave equation into

$$u_{\xi\eta} = 0. \tag{2}$$

(b) Solve (2) to deduce that the general solution of the 1-D wave equation is

$$u(x, t) = f(x - ct) + g(x + ct),$$

where $f$ and $g$ are arbitrary functions which are twice continuously differentiable.

(c) If the initial conditions are

$$u(x, 0) = \psi(x), \quad u_t(x, 0) = 0,$$

show that the general solution becomes

$$u(x, t) = \frac{1}{2}\left[\psi(x - ct) + \psi(x + ct)\right].$$

(d) Given that $c = 1$ and the initial condition is given by

$$\psi(x) = \begin{cases} 1 & \text{if } |x| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

sketch the solution at $t = 0$, $t = 1/2$, $t = 1$, $t = 2$. Describe what is happening.

5. Consider the three-dimensional wave equation given by

$$u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz}) \qquad -\infty < x, y, z < \infty, \; t > 0, \tag{3}$$

where $c$ is a real constant. We wish to study *spherical* waves of this equation (as would be observed, for example, as the waves propagating from a sound source in space).

(a) Write the equation in spherical polars and use radial symmetry to show that (3) becomes

$$(ru)_{tt} = c^2(ru)_{rr}, \qquad 0 < r < \infty, \; t > 0. \tag{4}$$

(b) Use the results of Problem 4 to show that the general solution of (4) is given by

$$u(r, t) = \frac{1}{r}\left[f(r - ct) + g(r + ct)\right].$$

6. Solve the following partial differential equations

   (a) $u_{xy} = 0$.

   (b) $u_{xyz} = 0$.

   (c) $u_{xy} = a(x, y)$.

7. Solve the differential equation

$$u_{xx} + 5u_{xy} + 6u_{yy} = e^{x+y},$$

   by reducing it to a form similar to that in Problem 5(c).

   [Hint: Factor the differential operator into $(\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y})(\gamma\frac{\partial}{\partial x} + \delta\frac{\partial}{\partial y})$ where the constants $\alpha, \dots, \delta$ are to be found. Then define new variables $\xi$ and $\eta$ which are appropriately chosen linear combinations of $x$ and $y$ to simplify the left hand side.]

8. Find the partial differential equation satisfied by the two-parameter family of spheres

$$z^2 = 1 - (x - a)^2 - (y - b)^2.$$

   [What you need here is an equation connecting $z$, $z_x$ and $z_y$.]

9. Find particular solutions of the equation

$$u_x^2 + u_y^2 = 1,$$

   of the form $u = f(x) + g(y)$.

10. Find particular solutions of the equation

$$u_x u_y = 1,$$

   of the forms $u = f(x) + g(y)$ and $u = f(x)g(y)$.

   [Hint: For Problems 9 and 10, substitute the suggested forms and then use separation of variables ideas to get *all* particular solutions.]