M2AA2 - Multivariable Calculus. Problem Sheet 6
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1. Let
$$I = \int_S \frac{\boldsymbol{x}\cdot\boldsymbol{n}dS}{|\boldsymbol{x}|^3}.$$
Show that $I = 4\pi$ if S is the sphere $|\boldsymbol{x}| = R$ and that $I = 0$ if S bounds a volume that does not contain the origin ($\boldsymbol{x} = 0$).

   Show that the electric field, defined for $\boldsymbol{x} \neq \boldsymbol{a}$ by $\boldsymbol{E}(\boldsymbol{x}) = \frac{q}{4\pi\epsilon_0}\frac{\boldsymbol{x}-\boldsymbol{a}}{|\boldsymbol{x}-\boldsymbol{a}|^3}$, satisfies
$$\int_{S_1} \boldsymbol{E}\cdot d\boldsymbol{S} = \begin{cases} 0 & \text{if a} \notin \text{V} \\ \frac{q}{\epsilon_0} & \text{if a} \in \text{V} \end{cases}$$
where S_1 is a closed surface bounding a volume V, and where the electric charge q, and permittivity of free space, ϵ_0, are constants. This is Gauss's law for a point electric charge.

2. The vector field $\boldsymbol{F}(\boldsymbol{x})$ is given in cylindrical polar coordinates r, θ, z (for $r \neq 0$) by
$$\boldsymbol{F}(\boldsymbol{x}) = r^{-1}\,\boldsymbol{e}_\theta.$$
Evaluate $\nabla\times\boldsymbol{F}$ using the formula for curl in cylindrical polars. Calculate $\oint_C \boldsymbol{F}\cdot d\boldsymbol{s}$, where C is the circle $z = 0$, $r = 1$ and $0 \leq \theta \leq 2\pi$. Does Stokes's theorem apply? Why not?

3. Show that the unit basis vectors of cylindrical polar coordinates satisfy
$$\frac{\partial \boldsymbol{e}_r}{\partial\theta} = \boldsymbol{e}_\theta, \qquad \frac{\partial \boldsymbol{e}_\theta}{\partial\theta} = -\boldsymbol{e}_r, \tag{1}$$
all other derivatives of the three basis vectors being zero.

   Given that the gradient operator in cylindrical polars is
$$\boldsymbol{\nabla} = \boldsymbol{e}_r\frac{\partial}{\partial r} + \boldsymbol{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \boldsymbol{e}_z\frac{\partial}{\partial z},$$
use (1) to obtain expressions for $\boldsymbol{\nabla}\cdot\boldsymbol{A}$ and $\boldsymbol{\nabla}\times\boldsymbol{A}$, where $\boldsymbol{A} = A_1\boldsymbol{e}_r + A_2\boldsymbol{e}_\theta + A_3\boldsymbol{e}_z$.

4. The surface S encloses a volume in which the scalar field satisfies the Klein-Gordon equation
$$\nabla^2 u = m^2 u, \;\; \boldsymbol{x} \in \boldsymbol{R}^3,$$
where m is a real non-zero constant. Prove that u is uniquely determined if either u or $\partial u/\partial n$ is given on S.

5. Find all solutions of the two-dimensional Laplace equation $\nabla^2\phi = 0$ that can be written in the separable form $\phi(r,\theta) = R(r)\Theta(\theta)$, where r and θ are plane polar coordinates.

   Hence solve, for $r < a$, the following boundary value problem, assuming that $\phi(r,\theta)$ satisfies a reasonable physical condition at $r = 0$:
$$\nabla^2\phi = 0, \qquad \phi(a,\theta) = \sin\theta. \tag{2}$$
Find also the solution for $r > a$ that satisfies $\phi(r,\theta) \to 0$ as $r \to \infty$.

6. The scalar function ϕ is a function only of the radial coordinate r in $\boldsymbol{R}^3$. Use Cartesian coordinates and the chain rule to show that

$$\nabla\phi = \phi'(r)\frac{\boldsymbol{x}}{r}, \qquad \nabla^2\phi = \phi''(r) + \frac{2}{r}\phi'(r). \tag{3}$$

Find the solution of $\nabla^2\phi = 1$ in the region $r \le a$ that is bounded and satisfies $\phi(a) = 1$.

7. Show that within a closed surface S, not more than one solution of Poisson's equation $\nabla^2\phi = f$ satisfies the boundary condition

$$g\frac{\partial\phi}{\partial n} + \phi = 0$$

on S, where $g(\boldsymbol{x}) \ge 0$ on S.

Show that $\phi(\boldsymbol{x}) = x$ satisfies Laplace's equation and the above boundary condition with S being the unit sphere $|\boldsymbol{x}| = 1$. Deduce that the condition $g(\boldsymbol{x}) \ge 0$ on S cannot be omitted in the above uniqueness theorem.

8. The functions u and w are defined in a volume V. Show that

$$\int_V |\nabla w|^2 dV - \int_V |\nabla u|^2 dV = \int_V |\nabla(w-u)|^2 dV + 2\int_V \nabla u\cdot\nabla(w-u) dV.$$

If $u = w$ on the boundary of V and u is harmonic, show that

$$\int_V |\nabla w|^2 dV \ge \int_V |\nabla u|^2 dV.$$

9. The scalar field ϕ is harmonic in a volume V bounded by a closed surface S. Given that V does not contain the origin ($r = 0$), show that

$$\int_S \left(\phi\nabla\left(\frac{1}{r}\right) - \left(\frac{1}{r}\right)\nabla\phi\right)\cdot\boldsymbol{n}\, dS = 0.$$

Now let V be the volume given by $\epsilon \le r \le a$ and let S_1 be the surface $r = a$. Given that $\phi(\boldsymbol{x})$ is harmonic for $r \le a$, use the above result, in the limit $\epsilon \to 0$, to show that

$$\phi(0) = \frac{1}{4\pi a^2}\int_{S_1} \phi(\boldsymbol{x}) dS.$$

Deduce that if ϕ is harmonic (but not constant) in a general volume V, then it attains its maximum and minimum values of S.

10. If $\nabla^2\phi = f(\boldsymbol{x})$ in a volume V enclosed by a surface S and $\boldsymbol{x}_0$ is a point within V, show that

$$4\pi\phi(\boldsymbol{x}_0) = -\int_V \frac{f(\boldsymbol{x})}{|\boldsymbol{x}-\boldsymbol{x}_0|}dV + \int_S \left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_0|}\frac{\partial\phi}{\partial n}(\boldsymbol{x})\frac{\partial}{\partial n}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_0|}\right)\right) dS.$$

Deduce a corresponding formula for the case where $\boldsymbol{x}_0$ lies on S.

[Hint: The first part is in your notes so you can replicate that derivation. When $\boldsymbol{x}_0$ is on the boundary, punch out a semi-spherical region around it rather than a spherical one as is done when $\boldsymbol{x}_0$ is within V, and follow a similar procedure.]