M2AA2 - Multivariable Calculus. Problem Sheet 4
February 16, 2009. Prof. D.T. Papageorgiou

1. Verify the Stokes theorem for the vector field

$$\boldsymbol{F} = (3x - y,\ -\frac{1}{2}yz^2,\ -\frac{1}{2}y^2z),$$

where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$, so that C is a circle in the $x - y$ plane.

2. Using the substitution

$$u = y^2 - x^2, \quad v = 2xy$$

or otherwise, evaluate the integral

$$I = \int\int_S (x^2 + y^2)^3 dxdy$$

where S is the finite region in the first quadrant, bounded by the lines $x^2 - y^2 = 1$, $y^2 - x^2 = 1$, $xy = 1$ and $xy = 2$.

3. Find the unit vectors $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$, $\boldsymbol{e}_z$ of a cylindrical polar coordinate system (r, θ, z) in terms of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$. Solve for $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ in terms of $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$, $\boldsymbol{e}_z$. Represent the vector $\boldsymbol{F} = y\boldsymbol{i} + z\boldsymbol{j} + x\boldsymbol{k}$ in cylindrical coordinates in the form $\boldsymbol{F} = F_1\boldsymbol{e}_r + F_2\boldsymbol{e}_\theta + F_3\boldsymbol{e}_z$ and verify that $\nabla \cdot \boldsymbol{F} = 0$ (this follows immediately from the Cartesian form).

4. Find the scale factors h_1, h_2, h_3 for the parabolic cylinder coordinates (u, v, z) given in terms of the Cartesian coordinates (x_1, x_2, x_3) by

$$x_1 = \frac{1}{2}(u^2 - v^2), \qquad x_2 = uv, \qquad x_3 = z.$$

If $\phi \equiv \phi(u, v, z)$ is a scalar field expressed in terms of parabolic cylinder coordinates, find $\nabla^2\phi$ in terms of these coordinates.

5. If (r, θ, φ) are spherical polar coordinates, verify that $u = r\cos\theta$ and $u = r^{-2}\cos\theta$ are solutions of Laplace's equation $\nabla^2 u = 0$.

6. Laplace's equation in the plane given in terms of polar coordinates (r, θ) is given by

$$\nabla^2\phi \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0.$$

In each of the cases (i) $0 \le r \le 1$, and (ii) $1 \le r < \infty$, find the general solution of Laplace's equation which is single-valued and finite.

Solve also Laplace's equation in the annulus $a \le r \le b$ with boundary conditions

$$\begin{aligned} \phi = 1 \quad &\text{on} \quad \text{r = a} \quad \text{for all} \quad \theta, \\ \phi = 2 \quad &\text{on} \quad \text{r = b} \quad \text{for all} \quad \theta. \end{aligned}$$