M2AA2 - Multivariable Calculus. Assessed Coursework I
February 16, 2009. Prof. D.T. Papageorgiou

DUE February 23, 2009, BEFORE 2PM

1. Newton's law of gravitation states that the force between that a fixed particle Q with coordinates (ξ, η, ζ) and mass m exerts on a second particle P with position (x, y, z) and unit mass is given by (apart from the gravitational constant G)

$$\boldsymbol{F} = m\nabla\left(\frac{1}{r}\right),$$

where $r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$ is the distance between P and Q. Since the force can be expressed as the gradient of a function we say that the potential of the force in the case of two particle is $\Phi = m/r$.

(a) Consider now the force exerted on P by n points $Q_1, Q_2, \ldots, Q_n$ with masses $m_1, m_2, \ldots, m_n$. Show that the potential of the force exerted on P by this system is

$$\Phi = \sum_{k=1}^{n} \frac{m_k}{\sqrt{(x-\xi_k)^2 + (y-\eta_k)^2 + (z-\zeta_k)^2}}.$$

(b) Suppose next that instead of being concentrated at a finite number of points, the masses are distributed in space with continuous density $\mu(\boldsymbol{x})$ over a region R, or a surface S or a curve C.

State what units μ measured in for the three cases and argue that the potentials in each case are given by

$$V_R(x,y,z) = \int\int\int_R \frac{\mu(\xi,\eta,\zeta)}{r} d\xi d\eta d\zeta, \qquad V_S(x,y,z) = \int\int_S \frac{\mu}{r} dS,$$

$$V_C(x,y,z) = \int_C \frac{\mu}{r} ds.$$

(c) Find the potential at a point P with coordinates (x, y, z) due to a sphere with centre the origin and of unit radius and unit density.

(d) If the potential is known then the force is found by taking the gradient of the potential. Find the components of this gravitational force on a unit mass at position (x, y, z) due to mass distributed with unit density in a region R of three-dimensional space.

What observations can you make regarding these forces and their corresponding potentials when the point (x, y, z) is (i) outside the region R, and, (ii) inside the region R. Do these potentials and forces remain bounded?

(e) Consider now a spherical surface S of radius a and unit surface density. Take the centre of the sphere to be the origin and place the point P on the x-axis so that its coordinates are $(x, 0, 0)$.

i. Show by transforming to spherical polar coordinates and carrying out all integrations that the potential V is given by

$$V(x) = \frac{2\pi a}{x}(|x+a| - |x-a|), \qquad x \neq 0.$$

ii. Conclude that the potential at an external point is the same as if the mass of the surface were concentrated at $\xi = 0$, and that the potential at an internal point is constant.

iii. Is the potential continuous across the interface? What about the component of the force in the x-direction? If there are discontinuities, calculate the jump.

2. The region R is defined as the finite area enclosed by the lines $y = x/2$, $y = 0$ and the curve $x = 1+y^2$. Sketch the region R. Using the change of variables $x = u^2+v^2, y = uv$ show that the transformed region R in the u, v domain is a triangle: pay particular attention to the mapped positions of the vertices. Calculate the area of the region R.

Is the coordinate system defined by u, v orthogonal? (Justify your answer.)

3. (a) State without proof Stokes' theorem for a vector field $\mathbf{A}$ with continuous derivatives, an open surface S bound by a closed (non-intersecting) curve C.

Hence, or otherwise, evaluate

$$\int\int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n} dS$$

where $\mathbf{A} = (4x^2 + y - 3)\mathbf{i} + 5xy\mathbf{j} + (xz + z^2)\mathbf{k}$ and S is (i) the surface of the hemisphere $x^2 + y^2 + z^2 = 9$ in $z \geq 0$ and (ii) the paraboloid $z = 1 - (x^2 + y^2)$ in $z \geq 0$.

(b) State without proof the divergence theorem satisfied by a differentiable function $\mathbf{A}$ in a simply connected volume V bounded by a surface S.

Hence or otherwise evaluate

$$\int\int_S \mathbf{A} \cdot \mathbf{n} dS$$

where $\mathbf{A} = (4x, 4y + z^2, 2zy)$ and S is the surface of a cube $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$.