M2AA2 - Multivariable Calculus. Problem Sheet 1 Solutions
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1. (i) $\boldsymbol{A}\cdot\boldsymbol{r} = A_1x_1 + A_2x_2 + A_3x_3$, hence $\nabla(\boldsymbol{A}\cdot\boldsymbol{r}) = (A_1, A_2, A_3) = \boldsymbol{A}$.

   (ii) $r^n = |\boldsymbol{r}|^n = (x^2+y^2+z^2)^{n/2}$. First component of $\nabla(r^n)$ is

$$\frac{\partial r^n}{\partial x} = nr^{n-1}\frac{\partial r}{\partial x} = nr^{n-1}\frac{x}{r} = nr^{n-2}x.$$

   Similarly for the other components, therefore

$$\nabla r^n = nr^{n-2}\boldsymbol{r}.$$

   (iii) $\boldsymbol{r}\cdot\nabla(x+y+z) = x+y+z$, therefore, $\nabla(\boldsymbol{r}\cdot\nabla(x+y+z)) = (1,1,1)$.

2. (a) Consider the first component, i.e.

$$\frac{\partial(\phi\psi)}{\partial x} = \phi\frac{\partial\psi}{\partial x} + \psi\frac{\partial\phi}{\partial x},$$

   with similar results for the y and z components. Hence the result follows.

   (b) Consider the first component: $\frac{\partial f(r)}{\partial x} = f'(r)\frac{x_1}{r}$. Putting all components together gives $\nabla(f(r)) = \frac{f'(r)}{r}(x_1,\ldots,x_n) = \frac{f'(r)}{r}\boldsymbol{r}$.

   (c) Noting that $\nabla^2 f(r) = \boldsymbol{\nabla}\cdot\nabla f(r) = \boldsymbol{\nabla}\cdot(\frac{f'(r)}{r}\boldsymbol{r})$ we obtain

$$\begin{aligned}\nabla^2 f(r) &= \frac{f'(r)}{r}\boldsymbol{\nabla}\cdot\boldsymbol{r} + \boldsymbol{r}\cdot\nabla(f'(r)/r)\\ &= \frac{nf'(r)}{r} + \boldsymbol{r}\cdot\left[\left(\frac{f''(r)}{r} - \frac{f'(r)}{r^2}\right)\frac{\boldsymbol{r}}{r}\right]\\ &= = f'' - \frac{(n-1)}{r}f'.\end{aligned}$$

   (d) The equation can be written as $\frac{1}{r^{n-1}}(r^{n-1}f')' = 0$, which can be integrated twice to yield $f(r) = \frac{A}{r^n} + B$ where A and B are constants. When $n = 2$ a solution to Laplace's equation is $f(r) = 1/(x^2+y^2)$.

3. The required derivative is $\boldsymbol{p}\cdot(\nabla\phi)_{(1,1,2)}$ where $\boldsymbol{p} = (1,2,3)/\sqrt{14}$ is a unit vector in the direction $(1,2,3)$. Calculating gives $(1,2,3)/\sqrt{14}\cdot(6,1,4) = 20/\sqrt{14}$.

4. Let the zero level sets of the functions $\phi_1 = x^2 + 2y^2 - z^2 - 8$ and $\phi_2 = x^2 + y^2 + z^2 - 6$ represent the two surfaces. The normals to the surfaces at $P(1,2,1)$ are $\boldsymbol{n}_{1,2} = \nabla\phi_{1,2}|_{(1,2,1)}$, i.e. $\boldsymbol{n}_1 = (2,8,-2)$, $\boldsymbol{n}_2 = (2,4,2)$.

   The tangent to surface 1 at P is $(\boldsymbol{r} - \boldsymbol{r}_0)\cdot\boldsymbol{n}_1 = 0$ where $\boldsymbol{r}_0 = (1,2,1)$. Hence the equation is
$$x + 4y - z = 8.$$

   The required angle (call it θ) is the angle between the normals; $\boldsymbol{n}_1\cdot\boldsymbol{n}_2 = |\boldsymbol{n}_1|\,|\boldsymbol{n}_2|\cos\theta$. Hence $\theta = \cos^{-1}(4/3\sqrt{3})$.

5. Let $\phi = 3x^2 y\sin(\pi x/2) - z$ and hence
$$\nabla\phi = \left(6xy\sin(\pi x/2) + \frac{3\pi x^2 y}{2}\cos(\pi x/2),\ 3x^2\sin(\pi x/2),\ -1\right)$$
   is normal to the surface at any point. In particular $\boldsymbol{n} = (6,3,-1)$ is a vector normal to the surface at $(1,1,3)$. The equation of the tangent is $[(x,y,z)-(1,1,3)]\cdot(6,3,-1) = 0$, i.e. $6x+3y-z = 6$ is the required equation.

   The marble will roll in the direction of *maximum decrease* of $\phi(x,y,z) = 0$. This is the direction $\boldsymbol{u} = -\nabla\phi/|\nabla\phi|$ and when this is evaluated at $x = 1, y = 1/2$ we find $\boldsymbol{u} = -(3,3,-1)/\sqrt{10}$. Hence the direction of descent is southwest.

6. Let F_1 be the focus at $(-ae,0)$ and F_2 that at $(ae,0)$ and take $P(x,y)$ to be any point on the ellipse. Note also the vectors $\overrightarrow{F_1P} = (x+ae,y)$ and $\overrightarrow{F_2P} = (x-ae,y)$, and the normal to the surface at P which is given by $\boldsymbol{n} = (2x^2/a^2, 2y/b^2)$. We can now find the angle between $\overrightarrow{F_1P}$ and $\boldsymbol{n}$ to be
$$\cos^{-1}\left(\frac{\left(\frac{2x}{a^2}(x+ae) + \frac{2y^2}{b^2}\right)}{|\boldsymbol{n}|\sqrt{(x+ae)^2 + y^2}}\right).$$
   After some algebra and using the fact $b^2 = a^2(e^2-1)$, this angle can be shown to be
$$\cos^{-1}\left(\frac{2}{a|\nabla\phi|}\right).$$
   Now repeat for the angle between the vector $\overrightarrow{F_2P}$ and $\boldsymbol{n}$ and show that it gives the same result, hence the answer.

   Also, $F_aP + F_2P = 2a$ which is independent of x and y.

Physical interpretation. This example shows that if you shine a ray of light from one focus of an ellipse it will bounce off the elliptical wall and get reflected to the other focus. The physical interpretation is that when a ray of light hits a surface it is reflected from the surface along a line that subtends the same angle with the tangent to the surface as did the original ray.

7. (i) $\nabla\phi = r^2\nabla x + x\nabla r^2 = r^2(1,0,0) + 2x\boldsymbol{r}$.
   (ii) $\nabla\cdot(xr^2\boldsymbol{r}) = xr^2\nabla\cdot\boldsymbol{r} + \boldsymbol{r}\cdot\nabla xr^2 = 6xr^2$.
   (iii) Consider first component of $\nabla\times(f(r)\boldsymbol{r})$. This is

$$\frac{\partial}{\partial y}(zf(r)) - \partial\partial z(yf(r)) = \frac{yzf'(r)}{r} - \frac{yzf'(r)}{r} = 0.$$

   Similarly for the other components, therefore the answer is $\mathbf{0}$.

8. (i) $\nabla\cdot(\boldsymbol{u}\times\boldsymbol{v}) = \nabla\cdot(0,-z^3,z^2y) = 2yz$, and $\nabla\times\boldsymbol{u} = (0,2z,0)$, $\nabla\times\boldsymbol{v} = (0,0,0)$. Hence RHS is equal to LHS.
   (ii) $\psi\boldsymbol{u} = (x^2z^2+y^2z^2+z^4,0,0)$, $\nabla\psi = (2x,2y,2z)$, so $LHS = 2xz^2$ and the $RHS = (2x,2y,2z)\cdot(z^2,0,0) = 2xz^2$.

9. Since $x = uv$ and $y = 1/v$, we have $u = xy$ and $v = 1/y$. Hence the chain rule gives

$$\begin{aligned}\frac{\partial}{\partial x} &\to \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}\\ &= y\frac{\partial}{\partial u} = \frac{1}{v}\frac{\partial}{\partial u}\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial y} &\to \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}\\ &= x\frac{\partial}{\partial u} - \frac{1}{y^2}\frac{\partial}{\partial v} = uv\frac{\partial}{\partial u} - v^2\frac{\partial}{\partial v}\end{aligned}$$

Now we can calculate (subscripts will be used to denote partial derivatives, e.g. $f_x = \frac{\partial f}{\partial x}$ etc.):

$$f_x = \frac{1}{v}F_u, \quad f_{xx} = \frac{1}{v}\left(\frac{1}{v}F_u\right)_u = \frac{1}{v^2}F_{uu}, \quad f_y = uvF_u - v^2F_v \tag{1}$$

$$\begin{aligned}f_{yy} &= \left(uv\frac{\partial}{\partial u} - v^2\frac{\partial}{\partial v}\right)\left(uvF_u - v^2F_v\right)\\ &= u^2v^2F_{uu} - 2uv^3F_{uv} + 2v^3F_v + v^4F_{vv}\end{aligned}$$

$$f_{xy} = \frac{1}{v}\left(uvF_u - v^2F_v\right)_u = \frac{1}{v}\left(vF_u + uvF_{uu} - v^2F_{uv}\right)$$

Now substitute all these expressions into the equation and eliminate x, y in favour of u, v to get

$$v^2F_{vv} = 0 \quad \Rightarrow \quad F_{vv} = 0.$$

The general solution is

$$F(u,v) = \alpha(u)v + \beta(u) \quad \Rightarrow \quad f(x,y) = \frac{\alpha(xy)}{y} + \beta(xy),$$

where $\alpha(\cdot)$ and $\beta(\cdot)$ are arbitrary functions of integration that can be fixed once boundary conditions are provided.

10. Use the chain rule on $g(x,y,t) = f(xt,yt)$ to find

$$\frac{\partial g}{\partial t} = f_x(xt,yt)\frac{\partial}{\partial t}(xt) + f_y(xt,yt)\frac{\partial}{\partial t}(yt) = xf_x(xt,yt) + yf_y(xt,yt),$$

as required.

Now if $f(xt,yt) = t^n f(x,y)$, we can differentiate this w.r.t t and use the result above for the LHS, i.e.

$$xf_x(xt,yt) + yf_y(xt,yt) = nt^{n-1}f(x,y).$$

Now put $t = 1$ to obtain the result.

To obtain the last part, differentiate g_t with respect to t again to get

$$\begin{aligned} g_{tt} &= x(f_x(xt,yt))_t + y(f_y(xt,yt))_t \\ &= x\left[xf_{xx}(xt,yt) + yf_{xy}(xt,yt)\right] + y\left[xf_{xy}(xt,yt) + yF_{yy}(xt,yt)\right] \end{aligned}$$

Also $(t^n f(x,y))_{tt} = n(n-1)t^{n-2}f(x,y)$, and so equating these and setting $t = 1$ gives us what we want.

Note: This can be done for functions $f : \boldsymbol{R}^k \to \boldsymbol{R}$ also, in a nice compact form. Denote the function $f(\boldsymbol{x})$ where $\boldsymbol{x} = (x_1, \ldots, x_k)$, and define $g(\boldsymbol{x}, t) = f(\boldsymbol{x}t)$.

Differentiating w.r.t. t we get

$$g_t = x_1 f_{x_1}(\boldsymbol{x}t) + \ldots + x_k f_{x_k}(\boldsymbol{x}t) = \boldsymbol{x} \cdot \nabla f(\boldsymbol{x}t).$$

It follows immediately by setting $t = 1$ as before that for a *homogeneous* function $f(\boldsymbol{x}t) = t^n f(\boldsymbol{x})$ we have the compact formula

$$\boldsymbol{x} \cdot \nabla f(\boldsymbol{x}) = (n-1)f(\boldsymbol{x}),$$

and higher derivatives can be taken to get higher order identities also as done by you in $\boldsymbol{R}^2$.