M2AA2 - Multivariable Calculus. Problem Sheet 3
February 5, 2009. Prof. D.T. Papageorgiou

1. (a) Show that the centroid $(\overline{x}, \overline{y})$ of a closed simply connected region S in $\boldsymbol{R}^2$ is given by

$$\overline{x} = \frac{1}{2A} \oint_C x^2 dy, \qquad \overline{y} = \frac{1}{2A} \oint_C y^2 dx,$$

where A is the area of S and C is its boundary.
[Recall that the centroid of a plane region is its centre of mass when the region is regarded as a thin body of constant density.]

(b) Use the result above to verify that the centroids of the regions $S_1 = \{(x, y) : 0 \leq x^2 + y^2 \leq a^2\}$ and $S_2 = \{(x, y) : b^2 \leq x^2 + y^2 \leq a^2\}$, are given by $(\overline{x}, \overline{y}) = (0, 0)$.

2. If ϕ and ψ are scalar fields, show that $\nabla\phi \times \nabla\psi$ is a solenoidal field and verify that $\frac{1}{2}(\phi\nabla\psi - \psi\nabla\phi)$ is a vector potential for it.

3. Use the divergence theorem to show that

$$\int_V \nabla \times \boldsymbol{u} \, dV = \int_S \boldsymbol{n} \times \boldsymbol{u} \, dS,$$

where V is a volume in $\boldsymbol{R}^3$ bounded by a surface S with outward normal $\boldsymbol{n}$.
[Hint: Apply the divergence theorem to $\boldsymbol{F} = \boldsymbol{u} \times \boldsymbol{K}$ where $\boldsymbol{K}$ is a constant vecror in $\boldsymbol{R}^3$.]

4. Consider a solenoidal vector field $\boldsymbol{F}$ in a region V in $\boldsymbol{R}^3$ which is enclosed by a surface S. If ϕ is a scalar field that takes a constant value on the surface S, show that

$$\int_V \nabla\phi \cdot \boldsymbol{F} \, dV = 0.$$

5. Evaluate $\int_S \boldsymbol{r} \cdot \boldsymbol{n} \, dS$ where S is a closed surface enclosing a volume V.

6. Verify the divergence theorem for the case when $\boldsymbol{F} = (x, 0, 0)$ and V is the cube $|x| \leq a$, $|y| \leq a$, $|z| \leq a$.

7. The gravitational force field of a mass M is given by $\boldsymbol{F} = -\frac{GM\boldsymbol{r}}{r^3}$, where G is a constant. Show that the flux of $\boldsymbol{F}$ across a closed surface S is

(a) zero if the origin O lies outside S.

(b) $-4\pi GM$ if O lies inside S.

What does this result tell us about the value of $\nabla \cdot \boldsymbol{F}$ at O?

8. Use spherical polar coordinates

$$x_1 = r \sin\theta \cos\phi, \quad x_2 = r \sin\theta \sin\phi, \quad x_3 = r \cos\theta,$$

to evaluate the surface integrals

$$I_1 = \int_S x_1^2 \, dS, \qquad I_3 = \int_S x_3^2 \, ds,$$

where S is the spherical surface $r = a$, and verify that $I_1 = I_3$. Explain why the value obtained for I_1 is to be expected from symmetry considerations.