M2AA2 - Multivariable Calculus. Assessed Coursework II
March 9, 2009. Prof. D.T. Papageorgiou
**DUE March 19, 2009, BEFORE 2PM**

1. (a) Use the method of images to find the two-dimensional Dirichlet Green's function for the upper half plane, i.e. solve

$$\begin{aligned} \nabla^2 G &= \delta(\boldsymbol{x}-\boldsymbol{x}_0), \qquad -\infty < x < \infty,\ y > 0 \\ G(x,0) &= 0. \end{aligned}$$

(b) Use this Green's function to find explicitly the solution to the problem

$$\begin{aligned} \nabla^2 \phi &= 0, \qquad -\infty < x < \infty,\ y > 0 \\ \phi(x,0) &= \begin{cases} 1 & \text{if } |x| \le 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{1}$$

along with an appropriate decay condition at infinity.

(c) Use your solution to verify directly that the boundary condition (1) is satisfied (do this by considering the intervals $|x| < 1$, $x > 1$ and $x < -1$, separately), and that $\phi \to 0$ as $x^2 + y^2 \to \infty$.

(d) Show that the solution along the line $x = 0$ is given by

$$\phi(0,y) = \frac{2}{\pi}\tan^{-1}(1/y).$$

Physically, the solution found in (b) can represent the temperature in the upper half plane due to a strip heater fixed to the wall and maintained at constant unit temperature. Show that given an integer $N \ge 2$ the temperature along $x = 0$ is reduced by a factor of $N$ relative to that of the wall at the position

$$y_N = \frac{1}{\tan(\pi/2N)}.$$

Hence estimate at how many heater lengths from the wall we need to move to in order to see the temperature reduced by 99% of its wall value.

2. *Time-dependent heat flow in one dimension.* Consider the partial differential equation for the temperature $u(x,t)$ in a rod of length $L$:

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L,\ t > 0 \tag{2}$$

where $\kappa > 0$ is a constant (the thermal diffusivity of the material), subject to the boundary conditions

$$u(0,t) = 0, \qquad u(L,t) = 0,$$

(i.e. the ends of the rod are kept at fixed temperature). The initial condition at $t = 0$ is

$$u(x,0) = U_0, \qquad 0 \le x \le L,$$

where $U_0$ is a positive constant.

(a) Use separation of variables to find all solutions of the form $u(x,t) = X(x)T(t)$ and hence show that

$$u(x,t) = \sum_{n=1}^{\infty} s_n \sin \frac{n\pi x}{L} \exp(-n^2\pi^2\kappa t/L),$$

with $s_n$ to be found.

(b) Determine $s_n$ and give the general solution for $u(x,t)$. Fixing $\kappa = L = 1$ estimate how long it takes for the temperature to decrease by a factor of 2 from its initial value $U_0$.

3. There are some things wrong in the following reasoning. Find the mistakes and correct them.

In this problem we attempt to obtain the Fourier cosine coefficients of $e^x$:

$$e^x = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}. \tag{3}$$

Differentiating yields

$$e^x = -\sum_{n=1}^{\infty} \frac{n\pi}{L} A_n \sin \frac{n\pi x}{L},$$

which is the Fourier sine series of $e^x$. Differentiating again yields

$$e^x = -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 A_n \cos \frac{n\pi x}{L}. \tag{4}$$

Since equations (3) and (4) give the Fourier cosine series of $e^x$, they must be identical. Thus we find

$$A_0 = 0, \quad A_n = 0,$$

which is obviously wrong!

By correcting the mistakes you should be able to obtain $A_0$ and $A_n$ *without* having to compute Fourier coefficients in the usual way. In any case, find the coefficients.

What would happen if instead of a cosine series we tried to construct a sine series of $e^x$ in the interval $[0, L]$? [Sketch the functions to see what is going on.]