M2AA2 - Multivariable Calculus. Problem Sheet 2. Solutions.
February 16, 2009. Prof. D.T. Papageorgiou

1. Consider the first component $\frac{\partial}{\partial y}(2xz + y^2) - \frac{\partial}{\partial z}(2yz + x^2) = 2y - 2y = 0$; similar calculations give 0 for the other two components.

   If $\boldsymbol{v} = \nabla\phi$ then $\frac{\partial \phi}{\partial x} = 2xy + z^2$; integrate to get $\phi(x, y, z) = x^2y + xz^2 + f(y, z)$ with f to be found. Same procedure integrating ϕ_y and ϕ_z to finally arrive at $\phi = x^2y + xz^2 + y^2z + A$ with A a constant. Since $\phi(\mathbf{0}) = 0$, this gives $A = 0$.

2. Functions are harmonic, therefore $\nabla^2\phi = \nabla^2\psi = 0$; level surfaces are orthogonal, therefore $\nabla\phi \cdot \nabla\psi = 0$. Compute

$$\begin{aligned}\nabla^2(\phi\psi) &= \nabla \cdot (\nabla(\phi\psi)) = \nabla \cdot (\phi\nabla\psi + \psi\nabla\phi) \\ &= \phi\nabla^2\psi + \nabla\phi \cdot \nabla\psi + \psi\nabla^2\phi + \nabla\phi \cdot \nabla\psi = 0\end{aligned}$$

3. Parametrise the straight line: $x = t$, $y = 2t$, $z = 3t$, $0 \le t \le 1$. Then

$$I = \int_0^1 (2trdt + 6t^2 2dt + 3t^2 3dt) = 23 \int_0^1 t^2 dt = 23/3.$$

4. (a) For the straight line parametrise $(x, y, z) = (2t, t, 3t)$, $0 \le t \le 1$, hence

$$I_{P_1} = \int_0^1 \Big(12t^2 2dt + (12t^2 - t)dt + 9tdt\Big) = 16.$$

   (b) Substitute the given parametrisation to find

$$I_{P_2} = \int_0^1 \Big(12t^2 4tdt + [4t^2(4t^2 - t) - t]dt + (4t^2 - t)(8t - 1)\Big)\, dt = \frac{71}{5}.$$

   (c) Similarly

$$I_{P_3} = \int_0^2 \left(3s^2 ds + (\frac{3}{4s^2} - \frac{s^2}{4})\frac{s}{2}ds + \frac{3s^2}{8}\frac{9s^2}{8}ds\right) = 16.$$

5.

$$\nabla \times \boldsymbol{F} = (-x + x, -y + y, -z + z) = \mathbf{0}.$$

   Now, $\phi_x = 2x - yz$, $\phi_y = -xz$, $\phi_z = 2z - xy$; integrating we find

$$\phi = x^2 - xyz + z^2 + A,$$

   where A is an arbitrary constant. The integral is independent of the path and hence

$$I = [\phi]_{(0,0,0)}^{(2,1,3)} = 9 - 6 + 1 = 4.$$

6. (a) On C_1 we parametrise by $x = \cos t$, $y = \sin t$, $0 \le t \le \pi/2$. It follows that $\int_{C_1}(F_1 dx + F_2 dy) = \int_0^{\pi/2}(1)dt = \frac{\pi}{2}$. On C_2 we have a straight line joining $(0,1)$ with $(1,0)$; the equation of the line is $y = -x+1$ so we can parametrise by $x = t$, $y = 1-t$ with $0 \le t \le 1$. The integral becomes

$$\int_0^1 \left(-\frac{1-t}{t^2+(1-t)^2}dt + \frac{t}{t^2+(1-t)^2}(-dt) \right) = -\int_0^1 \frac{dt}{2t^2-2t+1}$$

$$= -\frac{1}{2}\int_0^1 \frac{dt}{(t-1/2)^2+1/4} = -\left[\tan^{-1}\frac{(t-1/2)}{(1/2)}\right]_0^1 = -(\tan^{-1}(1) - \tan^{-1}(-1)) = -\frac{\pi}{2}$$

Hence the sum $\int_{C_1+C_2}\cdots = 0$.

(b) C_1 is parametrised as above BUT $0 \le t \le -3\pi/2$. The integrand is identical to what we had above, hence $\int_{C_1}\cdots = \int_0^{-3\pi/2}(1)dt = -3\pi/2$. The integral along C_2 is the same as above. Adding we get $\int_{C_1+C_2}\cdots = -2\pi$.

(c) C is parametrised by $x = \cos t$, $y = \sin t$ with $0 \le t \le 2\pi$ (not told direction so assume positive). Hence the answer is $\int_C \cdots = 2\pi$, i.e. negative that found in part (b).

$F_{1y} = F_{2x}$ as long as $(x,y) \ne \mathbf{0}$. Hence any closed curve not containing the origin will give an integral of zero as found in part (a), any closed curve that is negatively oriented will give an answer of -2π as found in part (b), and any closed positively oriented curve will give $+2\pi$ as found in part (c).

7. Calculate the area integral explicitly:

$$\int_0^b \int_0^a (2x_2 - a)dx_1 dx_2 = \int_0^b a(2x_2-a)dx_2 = ab(b-a).$$

Now calculate the path integral (anti-clockwise, +ve sense) Let P_1 be the path from $(0,0)$ to $(a,0)$; P_2 from $(a,0)$ to (a,b); P_3 from (a,b) to $(0,b)$; P_4 from $(0,b)$ to $(0,0)$. Calculate $\oint_P F_1\, dx_1 + F_2\, dx_2$ along each path:

- Along P_1: Parametrise by $x_1 = t$, $x_2 = 0$ with $0 \le t \le a$. On this path $\boldsymbol{F} = \mathbf{0}$, hence $\int_{P_1}\cdots = 0$.
- Along P_2: Parametrise by $x_1 = a$, $x_2 = t$ with $0 \le t \le b$. Hence $\boldsymbol{F} = (at, 2at)$ and $\int_{P_2}\cdots = \int_0^b 2at dt = ab^2$.
- Along P_3: Parametrise by $x_1 = a-t$, $x_2 = b$ with $0 \le t \le a$ since the path starts at (a,b) and decreases to the left to $(0,b)$. Here $F_1 = ab$, $F_2 = 2(a-t)b$, $dx_1 = -dt$, $dx_2 = 0$, hence $\int_{P_3}\cdots = \int_0^a ab(-dt) = -a^2 b$.

  Alternatively, we observe that $\int_{(a,b)}^{(0,b)} = -\int_{(0,b)}^{(a,b)}$ and can parametrise the path using $x_1 = u$, $x_2 = b$ with $0 \le u \le a$. This leaves $\int_{P_3}\cdots = -\int_0^a ab du = -a^2 b$ as before.
- Along P_4: Parametrise by $x_1 = 0$, $x_2 = b-t$, $0 \le t \le b$. Hence $\boldsymbol{F} = (a(b-t), 0)$ and $dx_1 = 0$, $dx_2 = -adt$. But since $F_2 = 0$ on P_4 we get $\int_{P_4}\cdots = 0$. (Can use the alternative $\int_{(0,b)}^{(0,0)} = -\int_{(0,0)}^{(0,b)}$ to arrive at the same conclusion.)

Now add these contributions to find $\int_{P_1+P_2+P_3+P_4}(F_1 dx_1 + F_2 dx_2) = ab(b-a)$ hence verifying Green's theorem in the plane.

8. Take $\boldsymbol{F} = \frac{1}{2}(-x_2, x_1)$ and apply Green's theorem to find

$$\text{Area} = \int\int_S dx_1 dx_2 = \frac{1}{2}\oint_C (x_1 dx_2 - x_2 dx_1).$$

Let P_1 be the path along the cycloid and P_2 that along the x_1-axis. See Figure 1 for a sketch of the paths and note that they are in the *clockwise*, i.e. negative, direction. Along P_2 the parametrisation is $x_1(t) = a(t - \sin t)$, $x_2(t) = a(1 - \cos t)$, $0 \le t \le 2\pi$. Hence,

$$\begin{aligned}
\frac{1}{2}\int_{P_1}(\ldots) &= \int_0^{2\pi} [a(t - \sin t)a \sin t - a(1 - \cos t)a(1 - \cos t)]\, dt \\
&= \frac{a^2}{2}\int_0^{2\pi} [t \sin t - 2 + 2\cos t]\, dt \\
&= \frac{a^2}{2}\left[-t\cos t + 3\sin t - 2t\right]_0^{2\pi} \\
&= -3\pi a^2.
\end{aligned}$$

Along P_2 we have $x_2 = 0$, hence $\int_{P_2} \ldots = 0$. Hence the area is $3\pi a^2$, the plus sign being necessary because we have to change the direction of the contour.

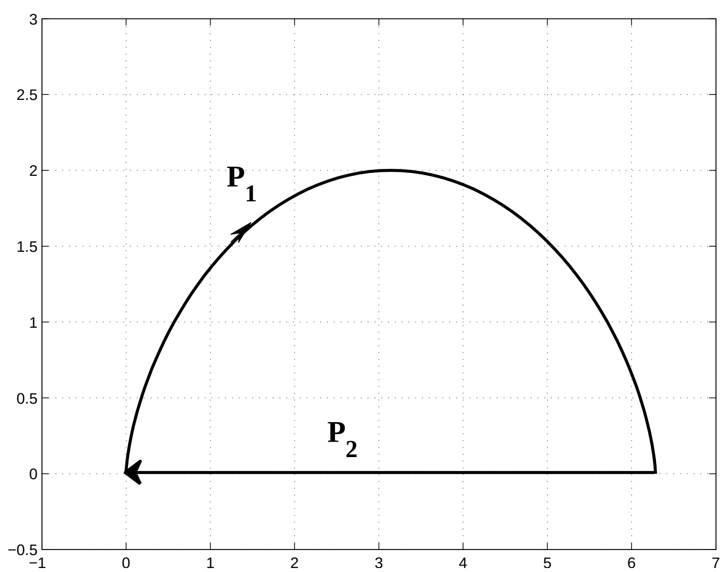

Figure 1: The contour used in problem 8.

The figure was produced using Matlab with the commands:

```
>> t=[0:2*pi/100:2*pi];
>> plot(t-sin(t),1-cos(t))
>> axis([-1 7 -.5 3])
>> grid
```