M2AA2 - Multivariable Calculus. Problem Sheet 9
March 24, 2009. Prof. D.T. Papageorgiou

1. If δ_{ij} is the Kronecker delta and x_i is a vector, evaluate
   (a) $\delta_{ij}\frac{\partial x_i}{\partial x_j}$.
   (b) $\delta_{ij}\delta_{ik}x_jx_k$.
   (c) $\delta_{ij}\frac{\partial^2\phi}{\partial x_i\partial x_j}$.
   (d) $\delta_{ij}\delta_{jk}\delta_{ki}$.

2. A particle of mass m and position $x_i$is in rigid body rotation with angular velocity ω_i relative to the origin O (that is its velocity is given by $\boldsymbol{v} = \boldsymbol{\omega}\times\boldsymbol{x}$). Show that the angular momentum h_i (given by $\boldsymbol{h} = m\boldsymbol{x}\times\boldsymbol{v}$) can be expressed as

$$h_i = I_{ij}\omega_j, \qquad \text{where} \qquad I_{ij} = m(\delta_{ij}x_kx_k - x_ix_j).$$

3. Use tensor notation to prove the identity

$$\boldsymbol{u}\times(\nabla\times\boldsymbol{u}) = \frac{1}{2}\nabla(|\boldsymbol{u}|^2) - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}.$$

4. (a) If u_i and v_i are tensors of rank one (i.e. vectors) show that $\phi = u_iv_i$ is a tensor of rank zero (i.e. scalar) and that $T_i = u_ku_kv_i$ is a tensor of rank one.
   (b) If u_i is a vector, show that $T_{ij} = \frac{\partial u_i}{\partial x_j}$ is a tensor.

5. A Cartesian coordinate system S' with orthogonal basis vectors $\boldsymbol{e}_1', \boldsymbol{e}_2', \boldsymbol{e}_3'$ is obtained from $S(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ by rotation through an angle θ about the $\boldsymbol{e}_3$ axis. Find the transformation matrix a_{ij} defined by $\boldsymbol{e}_i' = a_{ij}\boldsymbol{e}_j$.

   If $\boldsymbol{T}$ is a tensor with components T_{ij} relative to S, calculate T_{ij}' and verify that $T_{ii}' = T_{ii}$.

6. Let D_{ij} be defined as: $D_{ij} = 1$ if $i = j$ and $D_{ij} = 0$ otherwise, in any Cartesian coordinate system. Show that D_{ij} is not a tensor.

7. A tensor $\boldsymbol{T}$ of rank two has components T_{ij} given by

$$T_{11} = 4a, \qquad T_{23} = 4b, \qquad \text{other} \qquad T_{ij} = 0,$$

   with respect to a Cartesian coordinate system S.

   A coordinate system S' is obtained from S by rotation about the x_1-axis through an angle $\theta = +\pi/4$. A coordinate system S'' is obtained from S' by rotation through the x_3'-axis through an angle $\phi = +\pi/4$. Find the components of $\boldsymbol{T}$ in system S'' and verify the three identities

$$T_{ii}'' = T_{ii}, \qquad T_{ij}''T_{ij}'' = T_{ij}T_{ij}, \qquad T_{ij}''T_{ji}'' = T_{ij}T_{ji}.$$