M2AA2 - Multivariable Calculus. Problem Sheet 4. Solutions.
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1. Need to show that $\int\int_S (\nabla\times\boldsymbol{F})\cdot\boldsymbol{n}dS = \oint_C \boldsymbol{F}\cdot d\boldsymbol{s}$. First, calculate $\nabla\times\boldsymbol{F} = (0,0,1)$ and on S the normal is $\boldsymbol{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ (this is the outward pointing normal to S written in terms of spherical polars); therefore, $(\nabla\times\boldsymbol{F})\cdot\boldsymbol{n} = \cos\theta$. The integral (i.e. the l.h.s. of the theorem) becomes

$$\int\int_S (\nabla\times\boldsymbol{F})\cdot\boldsymbol{n}dS = \int_0^{2\pi}\int_0^{\pi/2}\cos\theta\sin\theta d\theta d\varphi = \int_0^{2\pi}\frac{1}{2}d\varphi = \pi.$$

The r.h.s. is a line integral over a circle of unit radius in the xy-plane. This is

$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{s} = \oint_C (3x-y)dx = \int_0^{2\pi}(3\cos\varphi-\sin\varphi)(-\sin\varphi)d\varphi = \left[-\frac{3}{2}\sin^2\varphi\right]_0^{2\pi} + \int_0^{2\pi}\sin^2\varphi d\varphi = \pi.$$

2. The required area is in the first quadrant and in terms of the new variables (u,v) it is the region $\overline{S}$ bounded by $-1\le u\le 1$, $1\le v\le 2$. The only other thing needed is the Jacobian of the transformation, which we compute to be

$$\frac{\partial(u,v)}{\partial(x,y)} = -4(x^2+y^2) \qquad \Rightarrow \qquad dxdy = \frac{dudv}{4(x^2+y^2)},$$

therefore

$$I = \int\int_{\overline{S}}\frac{(x^2+y^2)^2}{4}dudv = \int_2^4\int_{-1}^1\frac{(u^2+v^2)}{4}dudv = \int_2^4\frac{1}{4}(\frac{2}{3}+2v^2)^2dv = \frac{29}{3}.$$

3. We have

$$\boldsymbol{e}_r = \cos\theta\,\boldsymbol{i} + \sin\theta\,\boldsymbol{j}, \quad \boldsymbol{e}_\theta = -\sin\theta\,\boldsymbol{i} + \cos\theta\,\boldsymbol{j}, \quad \boldsymbol{e}_z = \boldsymbol{k}.$$

Solve to get

$$\boldsymbol{i} = \cos\theta\,\boldsymbol{e}_r - \sin\theta\,\boldsymbol{e}_\theta, \quad \boldsymbol{j} = \sin\theta\,\boldsymbol{e}_r + \cos\theta\,\boldsymbol{e}_\theta, \quad \boldsymbol{k} = \boldsymbol{e}_z.$$

Now, since $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ we get

$$\boldsymbol{F} = (r\sin\theta\cos\theta + z\sin\theta)\,\boldsymbol{e}_r + (-r\sin^2\theta + z\cos\theta)\,\boldsymbol{e}_\theta + r\cos\theta\,\boldsymbol{e}_z.$$

Now calculate $\nabla\cdot\boldsymbol{F}$ using the formula for cylindrical polars, to confirm that it is zero.

4. We have

$$dx_1 = udu - vdv, \qquad dx_2 = vdu + udv, \qquad dx_3 = dz,$$

so that

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 = (u^2+v^2)du^2 + (u^2+v^2)dv^2 + dz^2 \equiv h_1^2du^2 + h_2^2dv^2 + h_3^2dz^2.$$

Hence we identify the scale factors as

$$h_1 = \sqrt{u^2 + v^2}, \qquad h_2 = \sqrt{u^2 + v^2}, \qquad h_3 = 1.$$

We know that

$$\begin{aligned}\nabla^2 \phi &= \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial u} \left(\frac{h_2 h_3}{h_1} \frac{\partial \phi}{\partial u} \right) + \left(\frac{h_1 h_3}{h_2} \frac{\partial \phi}{\partial v} \right) + \left(\frac{h_1 h_2}{h_3} \frac{\partial \phi}{\partial w} \right) \right) \\ &= \frac{1}{(u^2 + v^2)} \left(\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2} \right) + \frac{\partial^2 \phi}{\partial z^2}.\end{aligned}$$

5. The Laplacian in spherical polars (r, θ, φ) leaving out any φ dependence (since the solutions we are constructing have dependence $\phi(r, \theta)$ is

$$\nabla^2 \phi = \frac{1}{r^2 \sin\theta} \left(\frac{\partial}{\partial r} \left(r^2 \sin\theta \frac{\partial \phi}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \phi}{\partial \theta} \right) \right).$$

Now substitute $\phi_1 = r \cos\theta$ and $\phi_2 = \frac{1}{r^2} \cos\theta$ to show the result.

6. For the general solution in (i) $0 < r < 1$ which is finite we have

$$\phi = c_0 + \sum_1^\infty r^n (c_n \cos n\theta + s_n \sin n\theta),$$

while for (ii) $1 < r < \infty$ the solution which is finite and single-valued is

$$\phi = c_0 + \sum_1^\infty r^{-n} (c_n \cos n\theta + s_n \sin n\theta).$$

The constants c_i, s_i are different in each formula.

For the problem in $a < r < b$ we cannot throw the logarithm out now. In fact the general solution is of the form

$$\phi = c_0 + d_0 \log(r) + \sum_1^\infty r^{-n} (c_n \cos n\theta + s_n \sin n\theta)$$

and clearly all the c_i, s_i, $i \geq 1$ are zero since the solution is independent of θ (we know this because the boundary conditions do not have any θ dependence). To satisfy the boundary conditions we need

$$\begin{aligned}\phi(a, \theta) &= 1 \quad \Rightarrow \quad c_0 + \log(a) = 1, \\ \phi(b, \theta) &= 2 \quad \Rightarrow \quad c_0 + \log(b) = 2.\end{aligned}$$

Solve for c_0 and d_0 to find the solution

$$\phi = 1 - \frac{\log(a)}{\log(b/a)} + \frac{\log(r)}{\log(b/a)}.$$