M2AA2 - Multivariable Calculus. Problem Sheet 8
March 23, 2009. Prof. D.T. Papageorgiou

1. In each of the problems that follow, find a stationary function for the integral satisfying the given conditions at the end points of the interval:

   (a) $\int_1^2 \frac{1}{x^3}(y')^2 dx$, $y(1) = 0$, $y(2) = -3$.

   (b) $\int_0^1 [\frac{1}{2}(y')^2 + yy' + y' + y]dx$, $y(0) = 0$, $y(1) = 4$.

2. Find a stationary function for $\int_{x_0}^{x_1} (1/x)\sqrt{1 + (y')^2} dx$ which passes through the points (x_0, y_0) and (x_1, y_1).

3. (a) Let $I = \int_{x_0}^{x_1} f(x, y, y', y'', y''')dx$. Show that a stationary function for this integral must satisfy

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''} - \frac{d^3}{dx^3}\frac{\partial f}{\partial y'''} = 0.$$

   (b) For the integral $I = \int_{x_0}^{x_1} f(x, y, y', y'', \ldots, y^{(n)})dx$ show that the Euler equation is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''} - \ldots + (-1)^n \frac{d^n}{dx^n}\frac{\partial f}{\partial y^{(n)}} = 0.$$

4. Find the extremals for $\int_{x_0}^{x_1} \left[a(y')^2 + 2byy' + cy^2\right] dx$, where a, b, c are given continuously differentiable functions of x. Prove that Euler's equation is a linear differential equation of the second order. Why is is that when b is constant this constant does not enter into the differential equation?

5. Find a stationary curve for the integral $\int_0^1 [16y^2 - (y'')^2 + x^2]dx$ satisfying the conditions $y(0) = 0$, $y(1) = 4$, $y'(0) = -1$, $y'(1) = 2$.

6. Find Euler's equations so that the integral $I(\phi) = \int_{x_0}^{x_1} F dx$ is stationary for the following cases (here the function $\phi \equiv \phi(x, y)$):

   (a) $F = (\phi_{xx} + \phi_{yy})^2 = (\nabla^2\phi)^2$.

   (b) $F = (\nabla^2\phi)^2 + (\phi_{xx}\phi_{yy} - \phi_{xy}^2)$.

7. Find Euler's equations for the isoperimetric problem in which

$$\int_{x_0}^{x_1} \left[a(u')^2 + 2buu' + cu^2\right] dx$$

   is to be stationary subject to the condition

$$\int_{x_0}^{x_1} u^2 dx = 1.$$

8. Show that the geodesics on a cylinder are helices.

9. Use the method of constrained optimisation to show that the solution to the classical isoperimetric problem (i.e. given the length of a closed curve, maximise the area it encloses) is a circle.