M2AA2 - Multivariable Calculus. Problem Sheet 5
March 2, 2009. Prof. D.T. Papageorgiou

1. Consider the following curvilinear coordinate system defined in terms of the cartesian system:

$$x_1 = \frac{1}{2}(u^2 - v^2), \qquad x_2 = uv, \qquad x_3 = z,$$

with $u \geq 0$.

Find the scale factors h_1, h_2 and h_3 of the transformation and express the unit vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ of the curvilinear coordinate system in terms of the Cartesian basis vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$.

2. The following vector field $\boldsymbol{F}$ is given in terms of the curvilinear coordinate system (u, v, z) given above

$$\boldsymbol{F} = u(u^2 + v^2)^{3/2}\, \boldsymbol{e}_1 - v(u^2 + v^2)^{3/2}\, \boldsymbol{e}_2.$$

Show that

(a) $\nabla \cdot \boldsymbol{F} = 4(u^2 - v^2)$.

(b) $\nabla \times \boldsymbol{F} = -8uv\, \boldsymbol{e}_3$.

(c) Show that in cartesian form $\boldsymbol{F} = 4(x_1^2 + x_2^2)\, \boldsymbol{i}$, and hence confirm your answers to parts (a) and (b) by doing the calculations in cartesian coordinates.

3. The temperature T in the region $a \leq r \leq b$ (where $r^2 = x^2 + y^2 + z^2$) satisfies Laplace's equation $\nabla^2 T = 0$. The spherical surface at $r = b$ is insulated so that $\partial T / \partial r = 0$ at $r = b$. The inner surface is maintained at the steady temperature $T = T_0 z/a$ so that $T = T_0 \cos\theta$ at $r = a$, where (r, θ, φ) are spherical polar coordinates. Find T in the region $a \leq r \leq b$.

[Hint: From the results of problem 5 of sheet 4, we know that $r\cos\theta$ and $r^{-2}\cos\theta$ satisfy Laplace's equation.]

4. Show that the solution you found in question 3 above is unique.

5. A vector field $\boldsymbol{B}(\boldsymbol{x})$ is parallel to the normals to a family of surfaces $f(\boldsymbol{x}) = \textit{constant}$. Show that

$$\boldsymbol{B} \cdot (\nabla \times \boldsymbol{B}) = 0.$$

6. Consider the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + z^2 = 1$, where $a > \sqrt{2}$ and $b > \sqrt{2}$. Let S be the part of the surface of the ellipsoid defined by $0 \leq x \leq 1$, $0 \leq y \leq 1$ and $z \geq 0$. Find the surface element $\boldsymbol{n} dS$ of the surface S. The vector field $\boldsymbol{F}$ is defined by $\boldsymbol{F} = (-y, x, 0)$. Explain why $\int_S \boldsymbol{F} \cdot \boldsymbol{n} dS = 0$ in the case $a = b$. Find $\int_S \boldsymbol{F} \cdot \boldsymbol{n} dS$ in the case $a \neq b$.

7. Let $\boldsymbol{F}(\boldsymbol{x}) = (x^3 + 3y + z^2,\ y^3,\ x^2 + y^2 + 3z^2)$, and let S be the *open* surface

$$1 - z = x^2 + y^2, \qquad 0 \leq z \leq 1.$$

Use the divergence theorem (and cylindrical polar coordinates) to evaluate $\int_S \boldsymbol{F} \cdot \boldsymbol{n} dS$.

8. By applying the divergence theorem to the vector field $\boldsymbol{k} \times \boldsymbol{A}$, where $\boldsymbol{k}$ is an arbitrary constant vector and $\boldsymbol{A}(\boldsymbol{x})$ is a vector field, show that

$$\int_V \nabla \times \boldsymbol{A} dV = -\int_S \boldsymbol{A} \times \boldsymbol{n} dS,$$

where the surface S encloses the volume V.

Verify this result when S is the sphere $|\boldsymbol{x}| = R$ and $\boldsymbol{A} = (z, 0, 0)$ in cartesian coordinates.

9. A solid volume V has mass density $\rho(\boldsymbol{x})$. The total mass M of V is then given by $M = \int_V \rho(\boldsymbol{x}) dV$. The *centre of mass* of V is defined to be

$$\hat{\boldsymbol{x}} = \frac{1}{M} \int_V \boldsymbol{x} \rho(\boldsymbol{x}) dV.$$

Find the centre of mass of (i) the solid upper hemisphere $x^2 + y^2 + z^2 \leq R^2$, $z \geq 0$ with constant mass density $\rho(\boldsymbol{x}) = \rho_0$, and (ii) the cone which has the unit circle in the xy-plane as a base, its tip at $(0, 0, h)$ $(h > 0)$ and has a mass density $\rho(\boldsymbol{x})$ proportional to the distance from its base.

10. A fluid flow has the velocity vector $\boldsymbol{u} = (0, 0, z + a)$ in cartesian coordinates, where a is a constant. Calculate the volume flux of fluid flowing across the open hemispherical surface $r = a$, $z \geq 0$, and also the volume flux across the disc $z = 0$, $r \leq 0$. Verify that the divergence theorem holds.

[The volume flux of fluid across a surface S is defined to be $\int_S \boldsymbol{u} \cdot \boldsymbol{n} dS$ where $\boldsymbol{u}$ is the velocity field.]