M2AA2 - Multivariable Calculus. Problem Sheet 2
January 26, 2009. Prof. D.T. Papageorgiou

1. If $\boldsymbol{v} = (2xy + z^2,\ 2yz + x^2,\ 2xz + y^2)$ show that $\nabla \times \boldsymbol{v} = 0$ and find the potential ϕ such that $\boldsymbol{v} = \nabla\phi$, with $\phi = 0$ at the origin.

2. If ϕ and ψ are harmonic scalar fields (i.e. they satisfy Laplace's equation) and if the level surfaces of ϕ and ψ are mutually orthogonal, show that the product $\phi\psi$ is harmonic.

3. Evaluate the line integral
$$I = \int_P (xy\, dx \ +\ yz\, dy \ +\ xz\, dz),$$
where P is the straight line joining the starting point $A(0,0,0)$ and the end point $B(1,2,3)$.

4. Evaluate the line integral $W = \int_P \boldsymbol{F} \cdot d\boldsymbol{r}$ along the three given different paths joining $(0,0,0)$ and $(2,1,3)$, where $\boldsymbol{F} = (3x^2, 2xz - y, z)$:

   (a) P_1: straight line.
   (b) P_2: the curve defined by $(2t^2,\ t,\ 4t^2 - t)$ from $t = 0$ to $t = 1$.
   (c) P_3: the curve defined by $(s, \frac{s^2}{4}, \frac{3s^3}{8})$ from $s = 0$ to $s = 2$.

5. If $\boldsymbol{F} = (2x - yz,\ -xz,\ 2z - xy)$, show that $\nabla \times \boldsymbol{F} = 0$ and find a function ϕ such that $\boldsymbol{F} = \nabla\phi$. Hence evaluate the line integral $\int_P \boldsymbol{F} \cdot d\boldsymbol{r}$, where P is any path from $(0,0,0)$ to $(3,2,1)$.

6. Consider the functions $F_1(x,y) = -\frac{y}{x^2+y^2}$ and $F_2(x,y) = \frac{x}{x^2+y^2}$ and the line integral $I = \int_C (F_1 dx + F_2 dy)$. Find directly the values of I for the three different *closed* curves given below

   (a) $C = C_1 + C_2$, where C_1 is the circular arc (traversed positively, i.e. anti-clockwise) joining $(1,0)$ with $(0,1)$ and C_2 is the straight line joining $(0,1)$ with $(1,0)$.
   (b) $C = C_1 + C_2$, where C_1 is the circular arc traversed *negatively* joining $(1,0)$ with $(0,1)$ and C_2 is the straight line joining $(0,1)$ with $(1,0)$.
   (c) C is a circle of radius 1 and centre $(0,0)$.

   Explain your findings especially in the light of the fact that $F_{1y} = F_{2x}$ and the path independence theorem that is expected to apply.

7. Verify Green's Theorem in the plane, namely
$$\oint_P F_1\, dx_1 \ +\ F_2\, dx_2 = \int\int_R \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) dx_1\, dx_2,$$
for the special case in which R is the rectangle with corners $(0,0)$, $(a,0)$, (a,b), $(0,b)$, $F_1(x_1,x_2) = ax_2$ and $F_2(x_1,x_2) = 2x_1x_2$ and P is the boundary of R.

8. Use Green's Theorem in the plane to show that the area A inside a plane curve C may be written in the form
$$A = \frac{1}{2}\oint_C x_1\, dx_2 \ -\ x_2\, dx_1.$$
Hence find the area bounded by one arc of the cycloid $x_1 = a(t - \sin t)$, $x_2 = a(1 - \cos t)$, with $a > 0$ and $t \in [0, 2\pi]$, and the x_1-axis.