M2AA2 - Multivariable Calculus. Problem Sheet 1
January 15, 2009. Prof. D.T. Papageorgiou

1. If $\boldsymbol{A}$ is a constant vector field, calculate the gradients of the following scalar fields:

$$(i)\ \ \boldsymbol{A}\cdot\boldsymbol{r} \qquad (ii)\ \ r^n \qquad (iii)\ \ \boldsymbol{r}\cdot\nabla(x+y+z)$$

[Here $r=|\boldsymbol{r}|$ where $\boldsymbol{r}\in\boldsymbol{R}^3$.]

2. (a) Prove the identity $\nabla(\phi\psi)=\phi\nabla\psi+\psi\nabla\phi$.
   (b) Prove that $\nabla(f(r))=\frac{f'(r)}{r}\boldsymbol{r}$, where $f'=\frac{df}{dr}$ and $\boldsymbol{r}\in\boldsymbol{R}^n$.
   (c) Show that $\nabla^2 f(r)=f''+\frac{(n-1)}{r}f'$ where $f(r)$ is the function in item (b) above.
   (d) Find the solutions of the symmetric Laplacian in $\boldsymbol{R}^n$. What happens when $n=2$?

3. If $\phi=x^2y+z^2x$ and P is the point $(1,1,2)$, find the directional derivative of ϕ at P in the direction $(1,2,3)$.

4. Find the equation of the tangent plane to the surface $x^2+2y^2-z^2-8=0$ at the point $P(1,2,1)$. A second surface $x^2+y^2+z^2-6=0$ also contains the point P. Find the angle between the two surfaces at the point P.

5. Obtain the equation of the plane that is tangent to the surface $z=3x^2y\sin(\pi x/2)$ at the point $x=y=1$.

   Take North to be the direction $(1,0,0)$ and East to be the direction $(0,1,0)$. In which direction will a marble roll if placed on the surface at $x=1$, $y=\frac{1}{2}$?

6. The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ has foci at $F_1(-ae,0)$ and $F_2(ae,0)$ where e is the eccentricity. Show that for each point $P(x,y)$ on the ellipse the lines PF_1, PF_2 make equal angles with the tangent to the ellipse at P. [This implies that any light ray directed from one focus is reflected by a mirrored ellipse to the other focus. How does the time taken for the reflected path F_1PF_2 depend on P?]

7. If $\phi=xr^2$, $\boldsymbol{r}=(x,y,z)$ and $f(r)=$ arbitrary function of $r=|\boldsymbol{r}|$, evaluate

$$(i)\ \ \nabla\phi \qquad (ii)\ \ \nabla\cdot(\phi\boldsymbol{r}) \qquad (iii)\ \ \nabla\times(f(r)\boldsymbol{r})$$

8. If $\boldsymbol{u} = (z^2, 0, 0)$, $\boldsymbol{v} = (x, y, z)$ and $\psi = r^2$, verify the identities

   (a) $\nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot (\nabla \times \boldsymbol{u}) - \boldsymbol{u} \cdot (\nabla \times \boldsymbol{v})$.

   (b) $\nabla \cdot (\psi \boldsymbol{u}) = \nabla \psi \cdot \boldsymbol{u} + \psi \nabla \cdot \boldsymbol{u}$.

9. Show that the change of variables $x = uv$, $y = 1/v$ reduces the equation

$$x^2 \frac{\partial^2 f}{\partial x^2} - 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} + 2y \frac{\partial f}{\partial y} = 0, \qquad (1)$$

   to

$$\frac{\partial^2 F}{\partial v^2} = 0,$$

   where $f(x, y) = F(u(x, y), v(x, y))$. Hence find the general solution of (1).

10. Let $f(x, y)$ be a given function and let $g(x, y, t) = f(xt, yt)$. Show that

$$\frac{\partial g}{\partial t} = x f_x(xt, yt) + y f_y(xt, yt),$$

   where the subscript denotes the partial derivative, e.g. $f_x(x, y) = \frac{\partial f(x,y)}{\partial x}$.

   If $f(x, y)$ is homogeneous of degree n, i.e. $f(xt, yt) = t^n f(x, y)$ deduce Euler's theorem

$$x f_x(x, y) + y f_y(x, y) = n f(x, y).$$

   Show also that $x^2 f_{xx}(x, y) + 2xy f_{xy}(x, y) + y^2 f_{yy}(x, y) = n(n-1) f(x, y)$.