UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

BSc/MSci EXAMINATION (MATHEMATICS) MAY – JUNE 2008
This paper is also taken for the relevant examination for the Associateship

M2AA2 MULTIVARIABLE CALCULUS

DATE: Thursday, 15 May 2008 TIME: 2.00 pm–4.00 pm

Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (a) A vector function $\mathbf{F}$ is defined as

$$\mathbf{F} = (x + 2y + az, bx - 3y - z, 4x + cy + 2z)$$

is this function invertible when $c = 6, b = -2, a = 0$?

(b) Find the tangent plane to the surface

$$\frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4xz - yz = 5$$

at $(1, 1, 1)$.

(c) Let $\hat{\mathbf{p}}$ be a unit vector and

$$\frac{\partial \phi}{\partial p} = \hat{\mathbf{p}} \cdot \nabla \phi$$

be the directional derivative of ϕ with respect to $\hat{\mathbf{p}}$. In what direction from the point $(1, 1, 1)$ is the directional derivative of a function ψ defined as

$$\psi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4xz - yz - 5$$

a maximum? What is the magnitude of the maximum?

(d) Let P be the path connecting $(0, 0, 0)$ and $(1, 1, 1)$ following the parametric curve $x = y = t$, $z = t^2$ for $0 \leq t \leq 1$. Evaluate the integral

$$\int_P \mathbf{F}.d\mathbf{r}.$$

(e) Find values of a, b, c such that there exists a scalar potential ϕ with $\mathbf{F} = \nabla \phi$, and find ϕ. Using these values of a, b, c determine

$$\int_P \mathbf{F}.d\mathbf{r}$$

where P is the straight line connecting $(0, 0, 0)$ to $(1, 1, 1)$.

(f) If ϕ is any solution of Laplace's equation show that $\nabla \phi$ is both irrotational and solenoidal, i.e. the curl of $\nabla \phi$ and the divergence of $\nabla \phi$ are both zero. Show that the specific function ϕ found in part (e) is harmonic.

2. (a) State without proof Green's theorem in the plane where C is a simple closed curve in the plane enclosing an area (surface) A.

Using Green's theorem find an expression for the area of a closed curve. Using your expression find the area of a triangle with vertices at (x_1, y_1), (x_2, y_2) and (x_3, y_3) in the x, y plane. Show that the area is given by the determinant

$$A = \frac{1}{2}\det \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}.$$

(b) State without proof the divergence theorem satisfied by a differentiable function $\mathbf{F}$ in a simply connected volume V bounded by a surface S.

Using the divergence theorem evaluate

$$\int\int_S \mathbf{F} \cdot \mathbf{n} \; dS$$

where $\mathbf{F} = (x-yz, y+xz, xy-2)$ and S is the surface given by the intersection of two cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

Turn over...

3. (a) Consider the functional J:

$$J = \int_{x_1}^{x_2} f(y_{xx}, y_x, y, x) dx$$

and deduce the Euler equation that f must satisfy for this functional to take extremal values. Here

$$y_{xx} = \frac{d^2y}{dx^2}, y_x = \frac{dy}{dx}$$

and you may assume that both y and y_x are fixed at the end-points, $x = x_1$ and $x = x_2$.

(b) Consider the following problem posed in the (x, y) plane. Given two points x_1 and x_2 ($x_2 > x_1$) on the x-axis and an arc-length L ($\frac{\pi}{2}(x_2 - x_1) > L > x_2 - x_1$), find the shape of the curve of length L joining x_1 with x_2 which, together with the x-axis, encloses the maximal area. The curve L is assumed to lie entirely in the x, y plane.

You may use the functional

$$J = \int_{x_1}^{x_2} (y + \lambda\sqrt{1 + y_x^2}) dx.$$

Justify its use and comment upon any deficiencies or limitations that it may have.

4. (a) Consider the wave equation for $u(x,t)$

$$u_{tt} = u_{xx}.$$

By rewriting this as a system of two first order partial differential equations deduce the characteristics are $x - t =$constant and $x + t =$constant. Thence the general solution is

$$u(x,t) = \frac{1}{2}\left(f(x-t) + g(x+t)\right)$$

for arbitrary functions f and g.

(b) Show that the surface defined by $z = u(x,y)$ cuts the family of surfaces $f(x,y,u) = c$ (where c is an arbitrary constant) orthogonally provided u satisfies the first order partial differential equation

$$\frac{\partial f}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial u}{\partial y} = \frac{\partial f}{\partial u}.$$

Hence find the surface that cuts the family

$$\frac{u(x+y)}{(3u+1)} = c$$

orthogonally and passes through $x^2 + y^2 = 1, u = 1$.

Turn over...

