New problem sheet in the back.

Lyapunov functions:

equilibrium pt x_0 of vector field f., ODE $\frac{dx}{dt} = f(x)$

$\exists$ nbh U of x_0 such that $\exists\, V: U \to \mathbb{R}$

$$\frac{d}{dt} V(x(t)) \leq 0 \quad \text{and} \quad V(x) \geq 0 \quad \forall x \in U$$

solns of ODE $\qquad V(x) = 0$ iff $x = x_0$.

$\Rightarrow$ x_0 is Lyapunov stable

if also $\frac{dV(x(t))}{dt} < 0 \quad \forall x(t) \in U \setminus \{x_0\}$

$\Rightarrow$ x_0 is asymptotically stable.

some examples:

(1) $\dot{x} = -x \quad x \in \mathbb{R} \qquad x = 0$ equilibrium

from derivative $\Rightarrow$ asympt stable.

$\exists$ Lyapunov function $V(x) = x^2$ on $\mathbb{R}$

$$\frac{d}{dt} V(x) = \frac{d}{dt}(x^2) = 2x \cdot \frac{dx}{dt} = -2x^2 \leq 0$$

$V(x) \geq 0 \quad V(x) = 0$ iff $x = 0$

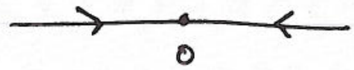
$\frac{d}{dt} V(x) < 0$ if $x \neq 0$

$\Rightarrow$ $x=0$ is asympt stable equilibrium

<u>and</u> its basin of attraction (set of initial condition y such that $\lim_{t \to \infty} x(t) = 0$ if $x(0) = y$) is $\mathbb{R}$

(there are many other Lyapunov fns for this problem!)

<u>other example:</u> $\dot{x} = -x^3$



$x=0$ is equilibrium, but derivative $=0$ $\Rightarrow$ non-hyperbolic!

linearization does not give any information about stability.

try as Lyapunov fc: $V(x) = x^2$

$$\frac{d}{dt} V(x) = 2x \cdot \frac{dx}{dt} = -2x^4 \leq 0$$

& same prop. as in previous example

$\Rightarrow$ $\underset{0}{\overset{x}{\shortparallel}}$ is asympt. stable with basin of attraction $= \mathbb{R}$

end of chap. 5.

In the remainder of this course we are going to focus a bit more on planar ODEs, because they are not so difficult to understand. This is because solution curves cannot "cross" due to existence & uniqueness. This hugely simplifies the flow.

For instance in $\mathbb{R}^3$ the situation gets very complicated sometimes: "spaghetti". $\Rightarrow$ "chaos"

In $\mathbb{R}^2$ there cannot be "chaos".

Chap 6 Limit sets (see also HSD chap 10)

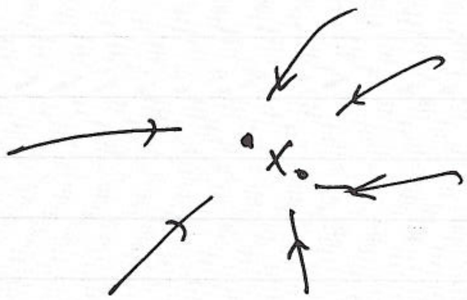
We have the following objective:

$\boxed{\text{Understand what happens to solns if } t \to \infty.}$ "dyn. systems theory"

Admittedly this objective is a bit arbitrary (because if $t >$ lifetime of the universe, who cares!)
Reasonable if dynamics is such that behaviour at $t \to \infty$ is "more or less" achieved already at much smaller times (e.g. it could be that we are attracted very rapidly to such limiting behaviour.)

For instance, if $\exists$ equilibrium x_0 that attracts all solutions $\Rightarrow$ answer is: equilibrium.

$$\lim_{t\to\infty} x(t) = x_0$$



Nice answer! ☺

This motivates the following definitions

ω-limit set

Let $x \in \mathbb{R}^m$ and ODE $\frac{dx}{dt} = f(x)$, with ϕ^t flow, then
$\omega(x)$ is defined to be the set of accumulation points of $\{\phi^t(x) \mid t \geq 0\}$ (positive semi-orbit of x under flow)

containing all pts $y \in \mathbb{R}^m$ such that $\exists$ mon. increasing sequence $\{t_k\}_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} t_k = \infty$ such that

$$\lim_{k\to\infty} \phi^{t_k}(x) = y$$

the ω-limit set "describes" what the point x converges to under the flow.

<u>Restate the objective:</u> to understand (the nature of) ω-limit sets.

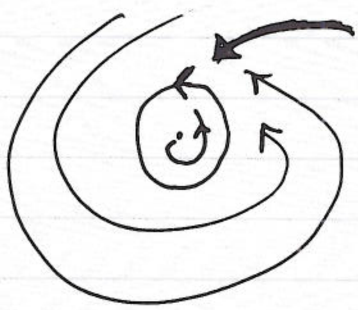
<u>Example:</u> if x_0 is an asymp. stable equilibrium pt with basin of attraction B then $\omega(y) = x_0$ for all $y \in B$.

(since by def. of B $\lim_{t\to\infty} \phi^t(y) = x_0$)

But ω-limit sets can be much more complicated!

eg. * periodic solution.

<u>Example:</u> recall our discussion of "Hopf bifurcation"



periodic solution is also an ω-limit set!

<u>observation:</u> if $x(t)$ is a periodic soln, then $\omega(y) = \{x(t)\}_{t\in\mathbb{R}}$ for all $y \in \{x(t)\}_{t\in\mathbb{R}}$

note but $\lim_{t\to\infty} x(t)$ does not <u>exist</u>!

In the above picture it is even suggested that the per soln is (globally) <u>asympt. stable</u> $\Rightarrow$ $\omega(y) = \{x(t)\}_{t\in\mathbb{R}}$ $\forall y \in \mathbb{R}^n$

"the distance between $y(t)$ and the periodic soln $\to 0$".

We can extend the definition of "asymptotic stability" to ω-limit sets.

Let $\underset{\cap \mathbb{R}^n}{A} = \omega(x)$ be an ω-limit set

then we can define the distance between any pt $y \in \mathbb{R}^n$

to A as
$$d(y, A) = \inf_{z \in A} d(y, z)$$
so that if $d(y, A) = 0 \Rightarrow y \in A$.

A is asymptotically stable if $\exists$ nbh U of A in $\mathbb{R}^n$ such that $\forall y \in U$ $\quad \lim_{t \to \infty} d(\phi^t(y), A) = 0$.

Could apply to periodic soln.

In the same way, we may define Lyapunov stability for ω-limit sets (or other invariant sets).

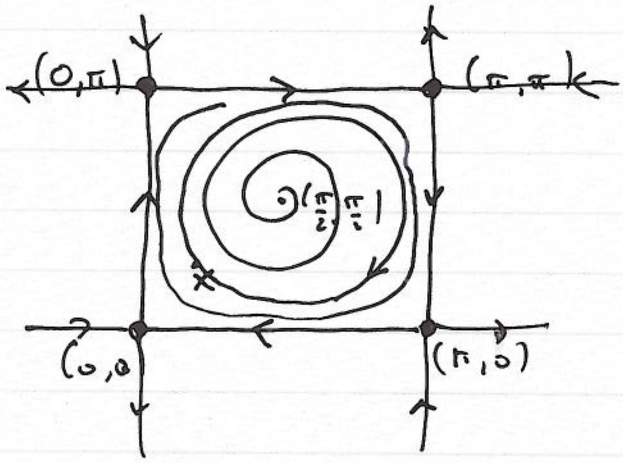
Question: what does an ω-limit set look like?
we know: possibly periodic soln, equilibrium

example:

$$\begin{cases} \dot{x} = \sin x\,(-0.1\cos x - \cos y) \\ \dot{y} = \sin y\,(\cos x - 0.1\cos y) \end{cases}$$

ODE in plane

part of phase portrait:



(exercise: proof!)

What is $\omega(x)$? It turns out that $(0,\pi)$, (π,π), $(\pi,0)$ and $(0,0)$ are all in $\omega(x)$

actually $\omega(x) = \{ \ell((0,0),(\pi,0)) \cup \ell((\pi,0),(\pi,\pi)) \cup \ell((\pi,\pi),(0,\pi)) \cup \ell((0,\pi),(0,0))$

where $\ell(a,b)$ denotes the line between a and b.

So $\omega(x)$ is a square!
Containing equilibria and connecting orbits between equilibria.

Theorem: (Poincaré - Bendixon)

ω-limit sets for autonomous ODEs in the plane are either: ($\mathbb{R}^2$)

- * equilibria
- * periodic solns
- * a "network" of equilibria connected by connecting orbits

Remark: such a result only in two dimensions. In higher dimension the situation is much more complicated (and unknown!): for instance $\omega(x)$ could be a "chaotic attractor".

Aim: (down the line), is to prove the P-B theorem and use it to analyze planar flows).

Some properties of ω-limit sets.

(1) if two points x, y lie on the same soln curve ($x = \phi^t(y)$ for some t)

then $\omega(x) = \omega(y)$

(2) if D is a closed, positively flow-invariant set and $x \in D$ then $\omega(x) \subset D$

$$\left(\phi^t(x) \in D \ \forall t \geq 0 \Rightarrow \text{all acc. pts of } \{\phi^t(x)\}_{t\geq 0} \text{ are in } D \right)$$

(3) $\omega(x)$ is closed (follows directly from def. : exercise)

We also define $\alpha(x)$ as the set of acc. pts of $\{\phi^t(x)\}_{t\leq 0}$ as $t \to -\infty$.

α-limit set of ϕ^t is ω-limit set of $(\phi^t)^{-1}$

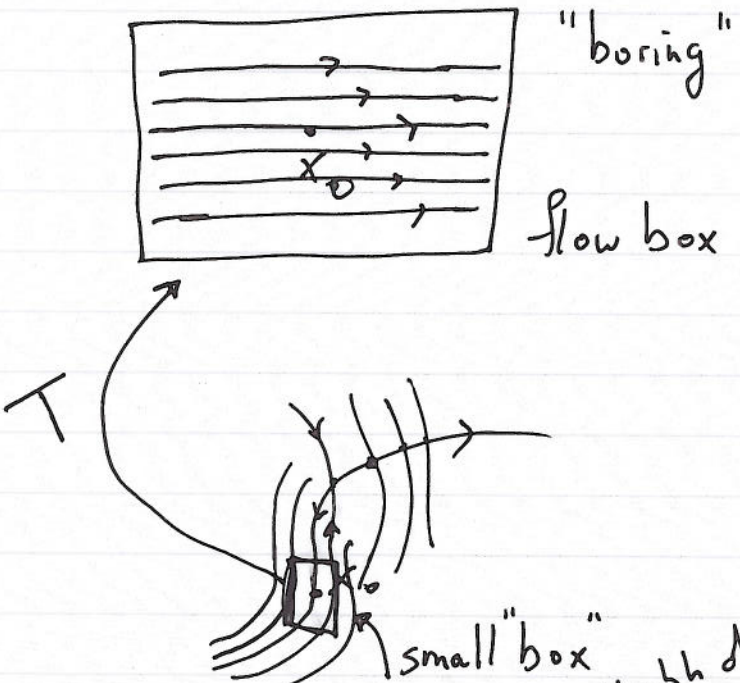
We would now like to discuss the nature of the flow <u>locally</u> near a single point. x_0.

Two situations:

(1) x_0 is an equilibrium pt → discussed before

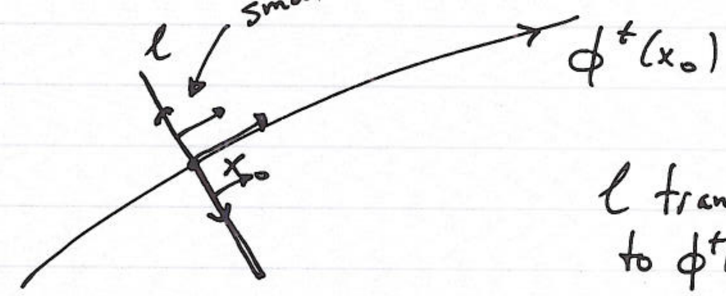
(2) x_0 is not " " "

In case (2) the situation is much easier. We are going to show this. Claim is that near x_0 (not eq.) the flow is boring (essentially a bunch of parallel motion, translation).



small nbh of x_0 in ℓ : U (local section)

<u>Construction:</u>



ℓ transverse to $\phi^t(x_0)$ at x_0.

Consider the flow of set of initial conditions in U.

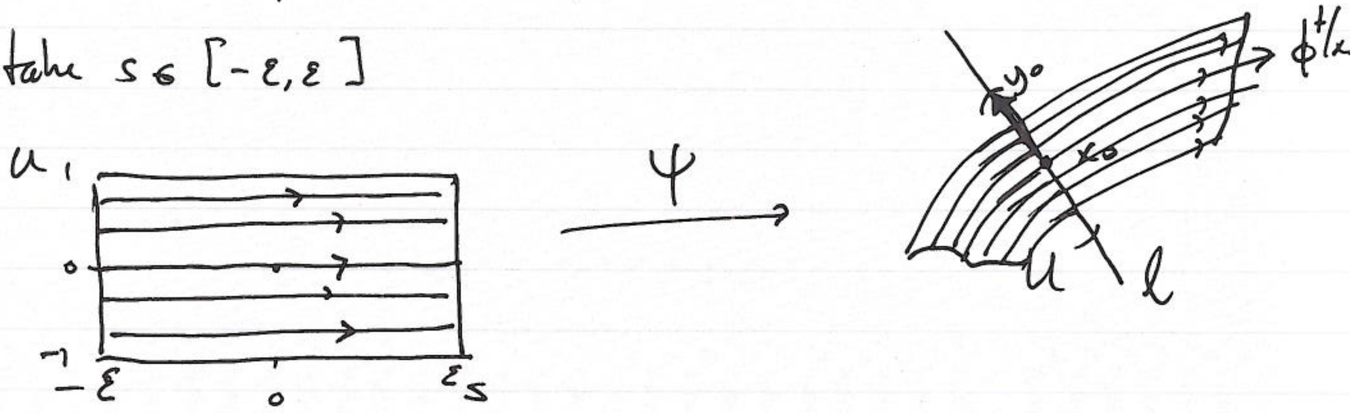
Observation: x_0 is <u>not</u> acc. by eq. pts (because otherwise, by continuity of flow, x_0 would also be eq pt) in U all vectors of the vector field are "close" to each other (by continuity)

Let U be labelled with coordinate u, and let s denote time, then we define.

→ points in U are written as $u \in [-1,1)$

$$\psi(s,u) = \phi^{s}(h(u)) \qquad h(u) = x_0 + u y_0$$

take $s \in [-\varepsilon, \varepsilon]$



ψ conjugates the local flow to the boring flow box

(smooth conjugacy)

Result: locally outside equilibria the flow is "easy".

The Poincaré map

Let x_0 be in $\mathbb{R}^m$ and <u>not</u> an equilibrium of $\dot{x} = f(x), x \in \mathbb{R}^m$

then let S be a <u>section</u> (also known as Poincaré section) at x_0 wrt. the flow ϕ^t of the ODE, i.e.

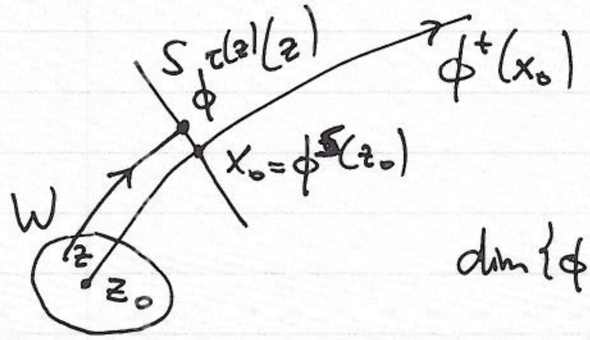
S is <u>transverse</u> to $\phi^t(x_0)$ at x_0. (x_0 is unique pt of intersection of $\phi^t(x_0)$ and S)

$\Downarrow$

$\dim S = m-1$

so that

$\dim\{\phi^t(x_0) \mid t \in [-\varepsilon, \varepsilon]\} + \dim S = \dim \mathbb{R}^m = m$



In case of $\mathbb{R}^2$, S is curve or line, in $\mathbb{R}^3$ S would be surface or plane.

Suppose z_0 is such that $\phi^s(z_0) = x_0 \in S$ $(s>0)$

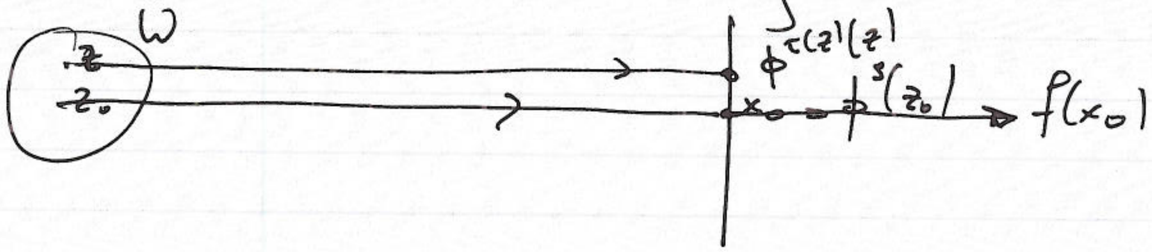
Let W be a nbh of z_0. We are interested in the intersection of $\phi^t(z)$ (with $z \in W$) and S.

Intuitively speaking, if W is small enough we would expect by continuity of the flow that $\forall z \in W$, $\exists$ time $\tau(z)$ near s such that

$$\phi^{\tau(z)}(z) \in S$$

It turns out that this is indeed the case, and we can furthermore establish the fact that $\tau : U \to \mathbb{R}$ is a differentiable fn (assuming as always that f, and hence also ϕ^t are differentiable).

Proof: (by application of the IFT)



We seek a formula that expresses the fact that $\phi^t(z) \in S$. We make some choices to simplify the argument.

Let $f(x_0)$ be the vector of the vector field f at x_0 and wlog S to be a perpendicular line to $f(x_0)$.

Let $y = (y_1, y_2) \in \mathbb{R}^2$ and $\eta : \mathbb{R}^2 \to \mathbb{R}$ the inner product

$\eta(y) = y \cdot f(x_0)$. Now the line S consists of points

$y = x_0 + v$ where $v \cdot f(x_0) = 0$

Hence this line is characterized by the fact

$$\eta(y) = (x_0 + v) \cdot f(x_0) = x_0 \cdot f(x_0)$$

Now define function: $G(z,t) = \eta(\phi^t(z)) = \phi^t(z) \cdot f(x_0)$

We know (by assumption, construction) that

$$G(z_0, s) = x_0 \cdot f(x_0) \Rightarrow \phi^s(z_0) \in S$$

Let $H = G - x_0 \cdot f(x_0) \Rightarrow H(z_0, s) = 0$

To apply IFT, we need to check that $\frac{\partial G}{\partial t}(z_0, s) \neq 0$ $\left(\Leftrightarrow \frac{\partial H}{\partial t}(z_0,s) \neq 0\right)$

$$\frac{\partial G}{\partial t}(z_0, s) = \underbrace{\frac{d}{dt}\phi^t(z)\Big|_{\substack{t=s\\ z=z_0}}}_{f(x_0)} \cdot f(x_0) = |f(x_0)|^2 \neq 0$$

since x_0 is not an equilibrium

for all z suff close to z_0

$$\Rightarrow \quad \frac{\partial H}{\partial t}(z_0, s) \neq 0 \Rightarrow \exists!\ \tau(z) \text{ with } \tau(z_0) = s$$

such that $H(z, \tau(z)) = 0$

$\Leftrightarrow \quad G(z, \tau(z)) = x_0 \cdot f(x_0)$

$\Leftrightarrow \quad \phi^{\tau(z)}(z) \in S$

by differentiability of H (and G) due to diff. of ϕ^t

we also find that $\tau : W \to \mathbb{R}$ is differentiable

qed

Differentiability of $\tau(z)$ is important since this implies that the map

$$h = \phi^{\tau(\cdot)}(\cdot) : W \to S \quad \text{is differentiable.}$$

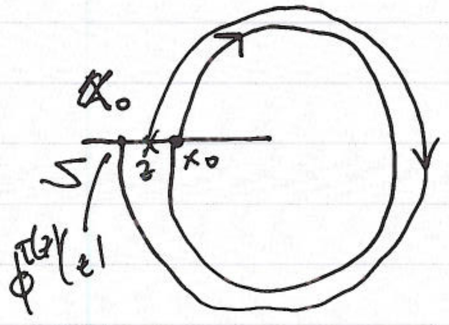
since ϕ^τ depends smoothly on argument ($\in W$) <u>and</u> τ.

hence ~~fld~~ $h(z) := \phi^{\tau(z)}(z)$ is differentiable.

We are about to use section to study the flow near ω-limit sets that are not equilibria.

In particular: let $\omega(x)$ be a periodic solution with period T

$\omega(x) = \{x(t) \mid t \in \mathbb{R}, x(0) = x_0\}$ $\quad (x(t) = x(t+T))$



We know: $\phi^T(x_0) = x_0 \in S$

If S is "short enough" (inside small nbh W of x_0) then we have seen that we can define a map

$$h : S \to S$$

such that $h(z) = \phi^{\tau(z)}(z) \in S$ and $\tau(z) > 0$ is the ~~shortest~~ smallest such that the time ~~[illegible]~~ $\tau(z)$ flow of $z \in S$ "first return time")

($\tau(z)$ is the time it takes z to return to S the first time after leaving S)

Note: it need not be the case that $\phi^{\tau(z)}(z) = z$!

Because of previous results, we know that h is nice!

h is differentiable and invertible (by existence & uniqueness)

Claim: h can tell me a lot about the stability properties of $\omega(x_0)$ (per. soln).

For instance: if $h: S \to S$ is a contraction then $\omega(x_0)$ is an asymptotically stable periodic soln.

Proof: h has a unique fixed pt in S, this is x_0! $h(x_0) = x_0$ (by construction) and all $z \in S$ converge exp. fast to x_0 under the map h. Then by continuity of the flow, all initial conditions y near $x(t)$ accumulate to this periodic solution $x(t)$.

in particular, if $|h'(x_0)| < 1$ then $\omega(x_0)$ is asymptotically stable

(by applicatn. of derivative test).