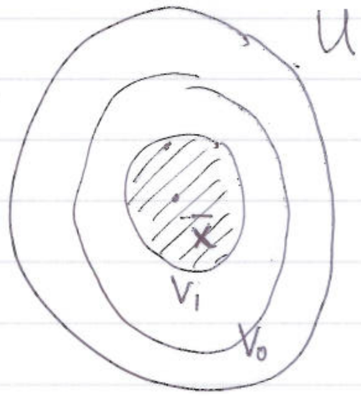
Lyapunov stability



$\boxed{} = V_1$

for every $y \in V_1$
the soln of the ODE
$x(t)$ with $x(0)=y$

has the property that

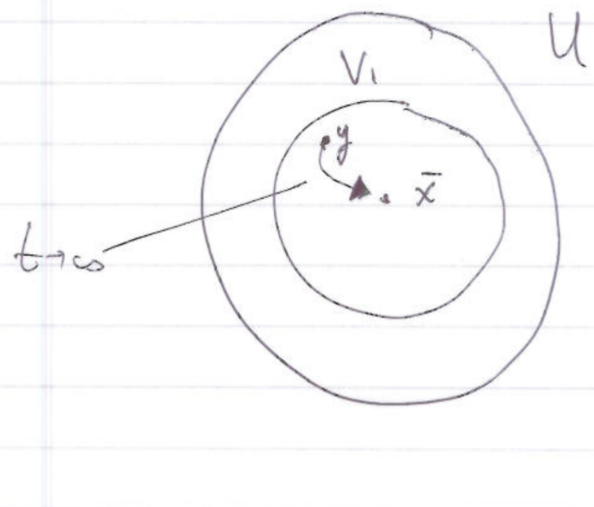
$$x(t) \in V_0 \quad \forall t \geq 0$$

$\forall U, \ \exists V_1, V_0$ with $V_1 \subset V_0 \subset U$.

In particular, $\forall$ nbh U of $\bar{x}$, $\exists$ nbh V_1 of $\bar{x}$ such that orbits starting in V_1 never leave U.

$\bar{x}$ is asymptotically stable if $\forall$ nbh U $\exists$ nbh V_1 of $\bar{x}$ such that all orbits starting in V_1 converge to $\bar{x}$ as $t \to \infty$

$\forall y \in V_1$ soln $x(t)$ with $x(0)=y$ satisfies $\lim_{t\to\infty} x(t) = \bar{x}$



Clearly if $\bar{x}$ is asympt stable $\Rightarrow$ $\bar{x}$ is Lyapunov stable.

example 1:

$$\frac{dx}{dt} = -x \qquad x \in \mathbb{R}$$

$$x(t) = e^{-t} x(0)$$

flow ϕ^t

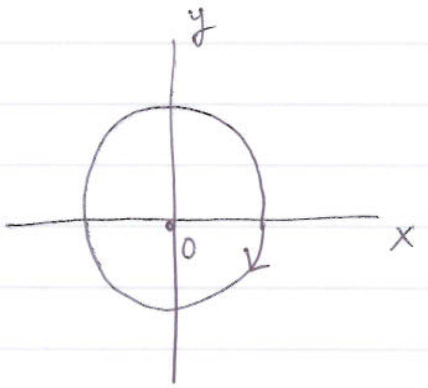
0 is an asymptotically stable equilibrium pt

example 2:

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -x \end{cases}$$

harmonic oscillator

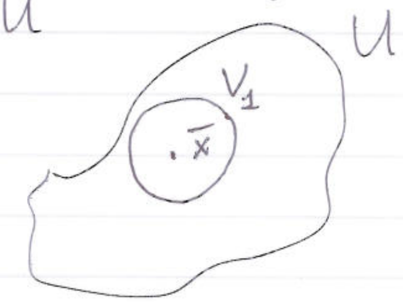
$$\text{flow } \phi^t = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$



all solns $x(t)$ are periodic $x(t) = x(t+2\pi)$ (period is 2π) and lie on circles.

not asympt stable since no soln (other then the equilibm itself) converges to 0

but 0 is Lyapunov stable. Namely, let V_1 be the smallest circular disk in U

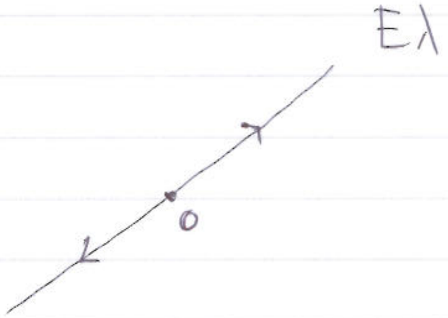


then no orbit starting in V_1 leaves V_1

if A has eigenval $\lambda \in \mathbb{R}$ $\quad \lambda > 0$

then the corresponding eigenspace E_λ is invariant under the flow $\quad \phi^t = \exp(At)$

$$\phi^t|_{E_\lambda} = e^{\lambda t}$$



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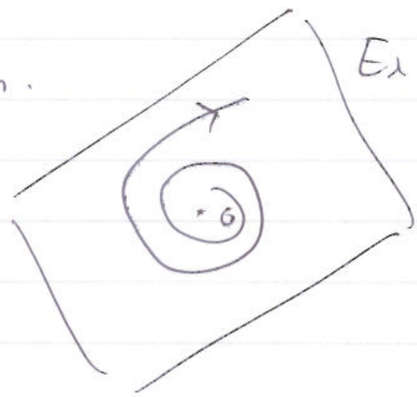
all solns $x(t)$ with
$x(0) = y \in E_\lambda \setminus \{0\}$

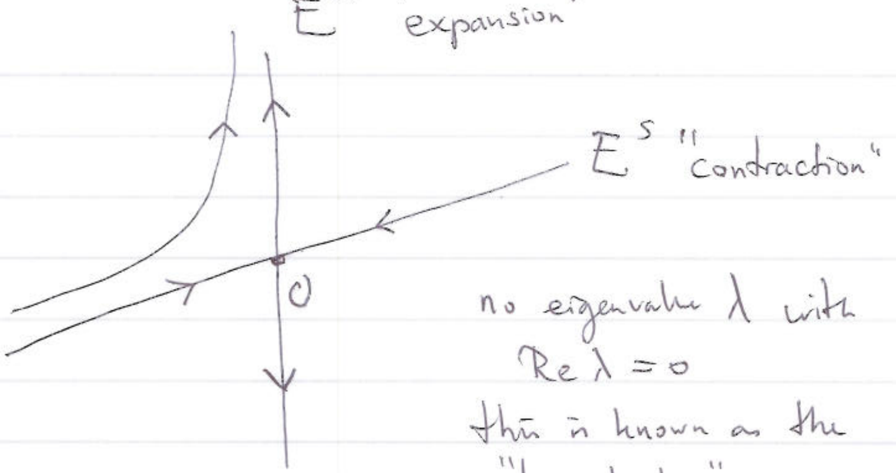
"set minus"

then $\lim_{t\to\infty} |x(t)| = \infty \quad \Rightarrow$ contradicts Lyap. stability..

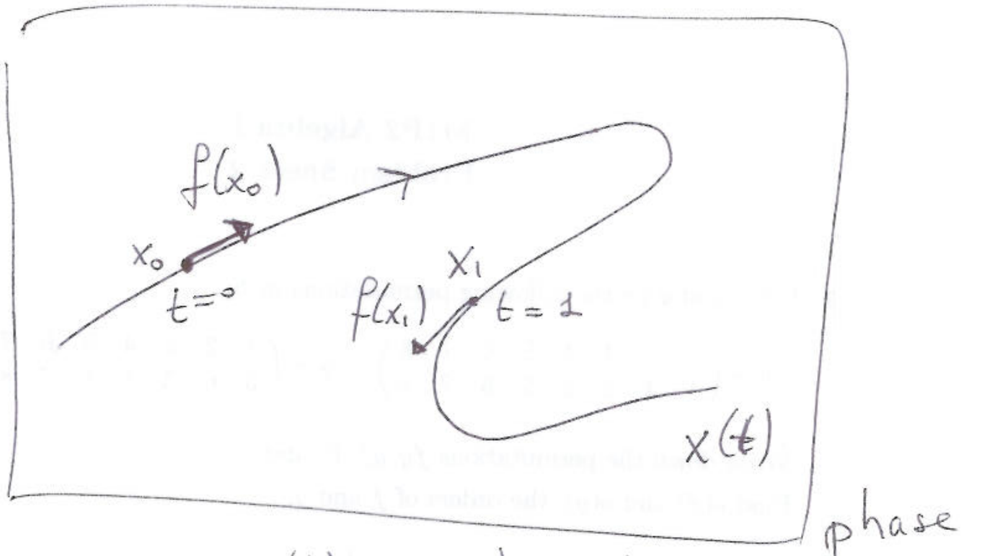
if $\lambda \notin \mathbb{R}$ but $\mathrm{Re}\,\lambda > 0$ then $\exists$ 2dim E_λ

such that the same prop holds.





no eigenvalue λ with
$\mathrm{Re}\,\lambda = 0$
this is known as the
"hyperbolic" case.



soln curves $x(t)$ are tangent to $f(x(\tilde{t}))$ at $x = \tilde{x}(\tilde{t})$

we can think of "solving ODE" as finding curves that are tangent to the vector field. by "existence and uniqueness" (proof later) through each pt in the phase space there is a unique curve with this property.

example of metric space

$\mathbb{R}^m$

with distance (Euclidean)

$$d(\vec{x},\vec{y}) = \sqrt{\sum_{i=1}^{m} (x_i - y_i)^2}$$

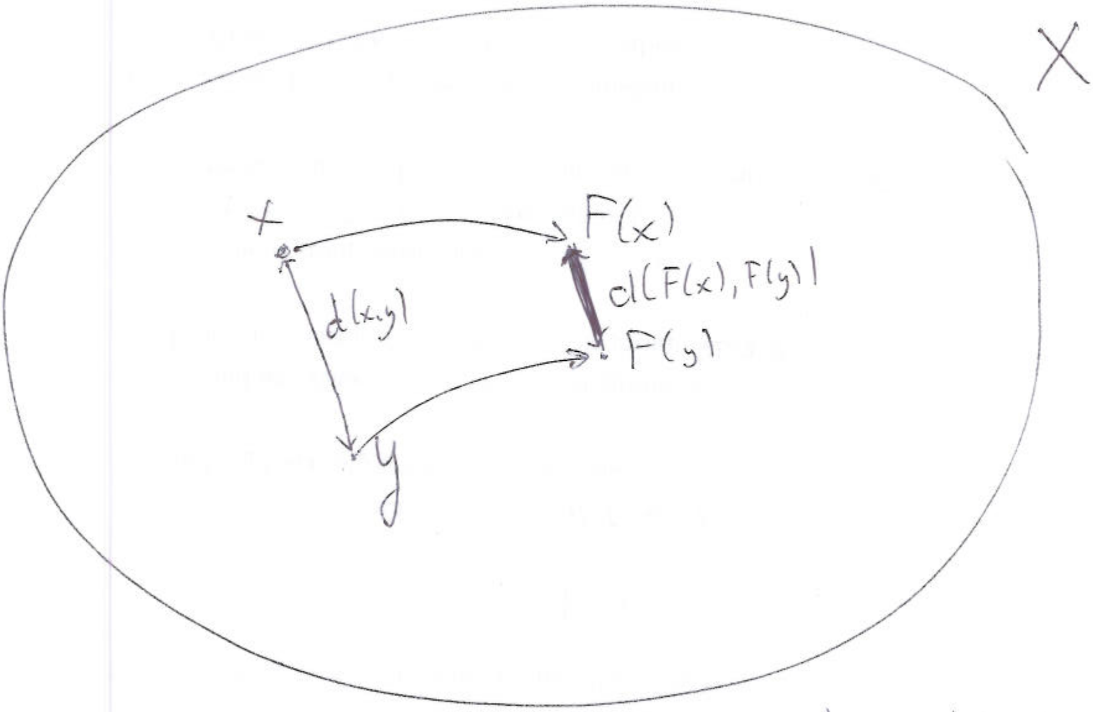
$\vec{x} = \sum_{i=1}^{m} x_i e_i$

$\vec{y} = \cdots$

e_i are orth. basis.

$(0,1)$ open interval

not a complete metric space.



warning: $d(F(x),F(y)) \le K\, d(x,y)$ $K<1$

$\cancel{\Uparrow}$ $\cancel{\Leftrightarrow}$ $\Downarrow$

$d(F(x),F(y)) < d(x,y)$

uniform contraction

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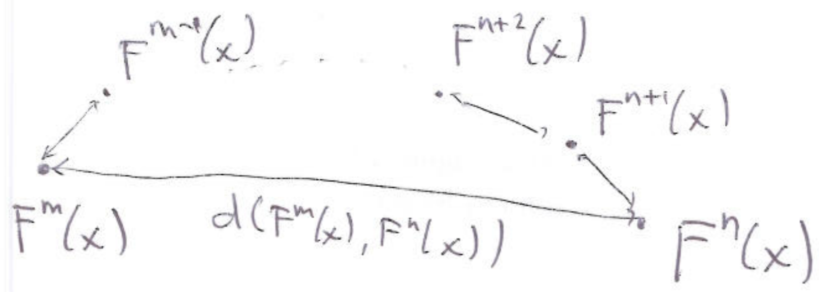
$m > n$



<u>example:</u> $F: \cancel{\mathbb{R}}\, I \to I \qquad I = [0,1]$

$$F(x) = \frac{1}{2}x$$

$$F'(x) = \frac{1}{2} < 1$$

$\Rightarrow$ F contraction

and F has unique fixed pt 0, $F(0) = 0$.

$\Phi^t : I \to I \qquad \phi^t = e^{-t}$ flow of $\frac{dx}{dt} = -x$

consider $\phi^t \quad t > 0$

then $(\phi^t)' = e^{-t} < 1$ if $t > 0$.

hence ϕ^t $(t>0)$ is contraction and

has unique fixed pt $\phi^t(0) = 0$

$$\lim_{t\to\infty} \phi^t(x) = \lim_{n\to\infty} (\phi^{t_0})^n(x) = 0 \qquad t_0 \text{ fixed} > 0$$