$$\frac{dx}{dt} = f(x,\lambda) \qquad x \in \mathbb{R}$$

$$f(x,\lambda) = \lambda + a x^2$$

equilibria $\quad x^2 = -\underbrace{\frac{\lambda}{a}}_{\text{if } <0}$ then no equilibria

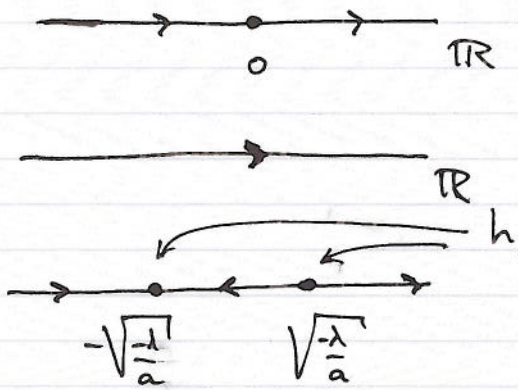
$a>0$ non-hyperbolic

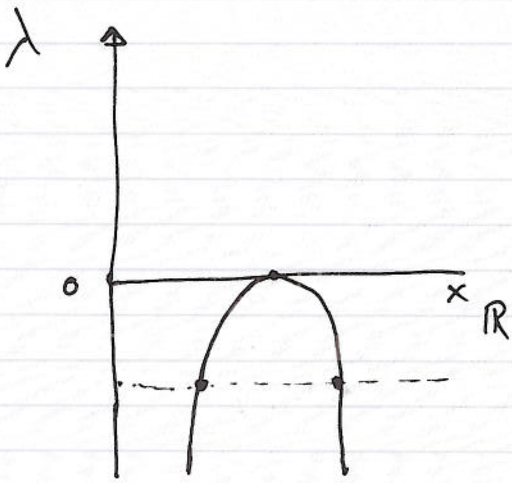
$\lambda = 0$

$\lambda > 0$

hyperbolic equilibria (check!)

$\lambda < 0$





curve of equilibria

"fold curve"

Question:

Is this type of bifurcation typical?

Let us consider, more generally

$$f(x,\lambda) = a(\lambda) + b(\lambda)x + c(\lambda)x^2 + O(x^3) \qquad x \in \mathbb{R}$$

(general smooth vector field)

Some assumptions: (1) $f(0,0) = 0$ equilibrium $x=0$ at $\lambda=0$

(2) $D_1 f(0,0) = 0$ equil. is non-hyperbolic

↑ "typical" in 1-par. family of 1D vector fields

(1) $\Leftrightarrow$ $a(0) = 0$

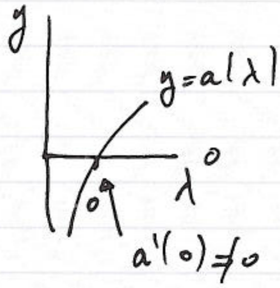
(2) $\Leftrightarrow$ $b(0) = 0$

We add another condition (3) $a'(0) \neq 0$ ($a(\lambda)$ passes 0 with "nonzero" speed)

$$\Rightarrow \operatorname*{sign}_{\lambda<0} a(\lambda) \neq \operatorname*{sign}_{\lambda>0} a(\lambda)$$

$|\lambda|$ suff small

This is a "typical" property of $a(\lambda)$ as it means that the graph of a intersects the line 0 transversely.



Condition (3) implies that

$$D_2 f(0,0) \neq 0$$

$$D_2 f(0,0) = a'(0)$$

Then by application of the IFT we have $!\,\lambda(x)$ with $\lambda(0)=0$ such that

$$f(x,\lambda(x)) = 0 \qquad \text{for } |x| \text{ suff small}$$

<u>Question</u>: what does $\lambda(x)$ look like?

<u>Aim</u>: find the Taylor expansion of λ as fn of x (up to certain order)

$f(x,\lambda) = a(\lambda) + b(\lambda)x + c(\lambda)x^2 + \dots$

let $a(\lambda) = a\lambda + O(\lambda^2)$

$\Rightarrow \quad \lambda \simeq -\dfrac{(\cancel{b(0)x} + c(0)x^2)}{a}$ As in simple example.

sloppy

how to do this more precisely: differentiate $f(x,\lambda) = 0$ wrt x at $x=0$, with $\lambda = \lambda(x)$

$$\frac{d}{dx} f(x,\lambda(x)) \Big|_{x=0} = \frac{d}{dx}\Big(a(\lambda(x)) + b(\lambda(x))x + c(\lambda(x))x^2 + O(x^3) \Big)\Big|_{x=0}$$

$$= \underbrace{a'(0)}_{\neq 0}\lambda'(0) + \cancel{b(0)} = 0 \quad \Rightarrow \boxed{\lambda'(0) = 0}.$$

now
$$\frac{d^2}{dx^2} f(x,\lambda(x))\Big|_{x=0} = \underbrace{a'(0)}_{\neq 0}\lambda''(0) + \cancel{b'(0)\underbrace{\lambda'(0)}_{=0}} + 2c(0) = 0$$

$$\lambda''(0) = -\frac{2c(0)}{a'(0)}$$

if $c(0) \neq 0$ then $\lambda''(0) \neq 0$

$\Rightarrow$ first term in Taylor exp of λ a fn of x is quadratic!

(4) if $c(0) \neq 0 \Rightarrow$ "quadratic fold"

if $c(0) = 0$ then maybe $\lambda \simeq c \cdot x^3$

?

Remember:

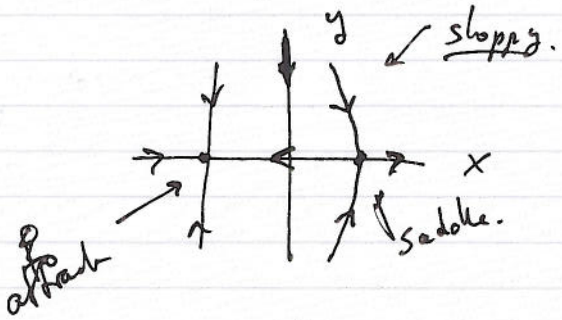
* can analyze "typical" systems
* use thm IFT
* approx soln by Taylor series obtained by diff of formula.

Conclusion: fold is typical bif in one-parameter families of 1-D vector fields.

Does this tell us anything about bifurcations in higher dimensional phase space?

Example:
$$\begin{cases} \dot{x} = \lambda + x^2 \\ \dot{y} = -y \end{cases}$$
ODE on $\mathbb{R}^2$

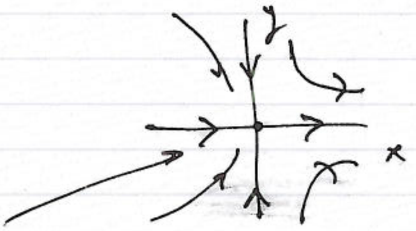
$\lambda < 0$

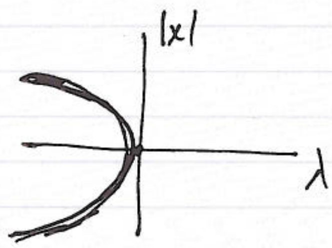


bif diagram is still "fold"

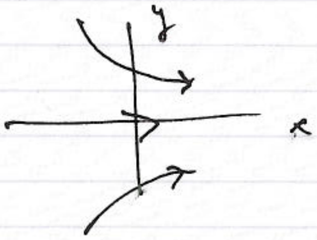
$\lambda = 0$

$D_1 f(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$





$\lambda > 0$



<u>Observation:</u> the bifurcation takes place in the "non-hyperbolic" direction.

<u>In fact:</u> <u>Theorem (no proof)</u>

Also if equilibrium is non-hyperbolic, there exist stable and unstable manifolds tangent to (and with the dimension of) E^s and E^u respectively.

<u>NB:</u> there actually also exist a "centre manifold" tangent to E^c on which all bifurcations take place. (here x-axis)

<u>Question:</u> in a one-parameter family of vector fields in $\mathbb{R}^m$ how can the set of equilibria typically change?

$$\dot{x} = f(x, \lambda) \quad x \in \mathbb{R}^m, \ \lambda \in \mathbb{R}$$

- suppose $\underset{x\ \ \lambda}{f(0,0)} = 0$ $\quad x=0$ equilibrium at $\lambda=0$.

- if $D_1 f(0,0)$ is invertible $\Rightarrow$ $x=0$ is hyperbolic and the set of equilibria cannot change (locally) if λ is changed because of IFT

$\exists! \, x(\lambda)$ s.t. $f(x(\lambda), \lambda) = 0$ with $x(0) = 0$.

- hence we need $D_1 f(0,0)$ to have a zero eigenvalue.

  <u>Assume:</u> $D_1 f(0,0)$ has exactly $\underline{\underline{1}}$ zero eigenvalue.

  (this is a persistent phenomenon in 1-par fam. of vector fields)

<u>Question:</u> how to proceed from here?
solving $f(x,\lambda) = 0$ near $x=0, \lambda=0$.

We will try using the IFT (implicit fn thm), by writing $f(x,\lambda)=0$ as <u>two</u> equations, one that can be solved by IFT and one that cannot be solved this way.

<u>Some observations:</u>

(1) $\ker D_1 f(0,0)$ has dimension 1 (spanned by eigenvector for eigenvalue 0)

we write:

(2) $\mathbb{R}^m = \ker D_1 f(0,0) \oplus C$

a complement to $\ker D_1 f(0,0)$ in $\mathbb{R}^m$

$C \simeq \mathbb{R}^{m-1}$

<u>direct sum:</u>

every vector in $\mathbb{R}^m$ can be written <u>uniquely</u> as a sum of one vector in $\ker D_1 f(0,0)$ and one in C

now introduce new coordinates $\mathbb{R}^m \ni x = (y, z)$

$y \in \ker D_1 f(0,0)$, $\quad z \in C$

---

(3) the Range $D_1 f(0,0)$ has dimension $m-1$

we can write

$$\mathbb{R}^m = \underbrace{\mathrm{Range}\, D_1 f(0,0)}_{\text{dimension } m-1} \oplus \underbrace{\tilde{C}}_{\text{complement to Range } D_1 f(0,0) \text{ in } \mathbb{R}^m}$$

$$\tilde{C} \simeq \mathbb{R}$$

(4) rewrite $f : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$

as
$$\begin{cases} f_1 : \ker D_1 f(0,0) \times C \times \mathbb{R} \to \mathrm{Range}\, D_1 f(0,0) \\ f_2 : \ker D_1 f(0,0) \times C \times \mathbb{R} \to \tilde{C} \end{cases}$$

$$f(x, \lambda) = 0 \iff \begin{cases} f_1(y, z, \lambda) = 0 \\ f_2(y, z, \lambda) = 0. \end{cases}$$

(5) The <u>important</u> observation now is that

$$D_2 f_1(0,0,0) \text{ is } \underline{\text{invertible}} \text{ by construction}$$

D_2: der. wrt z variable

hence we can use the IFT to solve for z as fn of y, λ so that $f_1(y, z(y,\lambda), \lambda) = 0$ with $z(0,0) = 0$

(6) it now remains to solve

$$g(y,\lambda) := f_2(y, z(y,\lambda), \lambda) = 0$$

$$g : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \quad \text{with } g(0,0) = 0 \quad D_1 g(0,0) = 0$$

<u>We know about this problem!</u>

(Lyapunov - Schmidt reduction)

$$f(\overbrace{x,y}^{\vec{x}},\lambda) = (\lambda + x^2, -y)^T$$

$$f(0,0,0)=0$$

$$J = D_{x,y} f(0,0,0) = \begin{pmatrix} \overset{0}{\cancel{2x}} & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathbb{R}^2 = \ker J \oplus C \qquad \ker J = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$$

$$\vec{x} \in \mathbb{R}^2, \quad \vec{x} = (x, y) \qquad \text{choose } C = \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$$

$$\mathbb{R}^2 = \text{Range } J \oplus \tilde{C} \qquad \text{Range } J = \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$$

$$\text{choose } \tilde{C} = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$$

$$f_2(x,y,\lambda) = \lambda + x^2 \qquad f_2 : \ker J \times C \times \mathbb{R} \to \cancel{\text{Range } J}\ \tilde{C}$$

$$f_1(x,y,\lambda) = -y \qquad f_1 : \ker J \times C \times \mathbb{R} \to \text{Range } J.$$

$(D_2 f_1(0,0,0) = -1)$

$$\exists!\ y(x,\lambda) \text{ solving } f(x, y(x,\lambda), \lambda) = 0$$

$$y(x,\lambda) = 0$$

$\Rightarrow$ we are left to solve: $g(x,\lambda) := f_2(x,0,\lambda) = \lambda + x^2$

(1 dim. problem).

<u>Conclusion</u>: typically in one-par families of ODEs locally the nr of equilibria changes as in a <u>fold</u> bifurcation.

(Note: in practice we cannot usually solve for $y(x,\lambda)$ ~~explicitly~~, but we can find / derive a Taylor series expansion for y as fc of x and λ, and this usually suffices.

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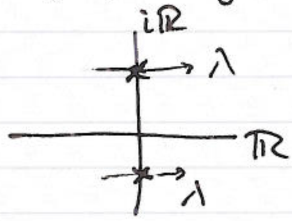
IV

We now consider what happens if the derivative of vector field at equilibrium point has a pair of purely imaginary eigenvalues



example:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

evals $\lambda \pm i$ $\quad$ if $\lambda = 0$ evals $\pm i$

if $\lambda < 0 \Rightarrow$ equilibrium $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is asympt. stable

$\lambda > 0 \Rightarrow$ " " " unstable

(indeed local flow changes considerably if λ passes through 0)

Q: What happens if the vector field is not linear?

(part of the) answer: also if the vector field is nonlinear, we see a change of stability of the equilibrium pt, if evals of Jacobian pass through $i\mathbb{R}\setminus\{0\}$

..... but is this the whole story?

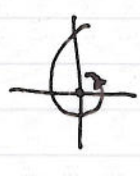
Example:
$$\begin{cases} \dot{x} = \lambda x - y - x(x^2+y^2) \\ \dot{y} = x + \lambda y - y(x^2+y^2) \end{cases}$$

in polar coordinates
$x = r\cos\theta$
$y = r\sin\theta$

$$\begin{cases} \dot{r} = \lambda r - r^3 \\ \dot{\theta} = 1 \end{cases}$$

$\dot{r} = 0$ iff $r(\lambda - r^2) = 0$
$r = 0$
$\lambda = r^2$

$\lambda < 0$



"Hopf-bifurcation"

Lyapunov functions

These are useful to establish Lyapunov and/or asymptotic stability of equilibria.

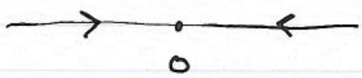
We of course already have information from derivative of vector field at eq. pt.

May be useful: (1) if eq. is not hyperbolic

(2) to obtain more info about the "region of stability" (i.e. basin of attraction in case of asymp. stable eq.)

Example

(ad 1) $\dot{x} = -x^3 = f(x) \quad x \in \mathbb{R}$



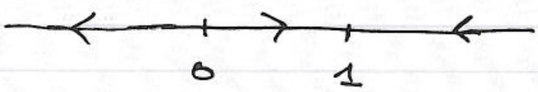
$f'(0) = 0$

$\Rightarrow 0$ is a non-hyperbolic equilibrium pt

all soln $x(t)$ satisfy

$$\lim_{t\to\infty} x(t) = 0$$

(ad2) $\dot{x} = x(1-x) = f(x) \quad x \in \mathbb{R}$ two equilibria $x=0, \ x=1$



$f'(1) = (1-2x)\big|_{x=1} = -1 \Rightarrow x=1$ is asympt stable eq. pt.

from picture we see all $x(t)$ with $x(0)>0$ satisfy $\lim_{t\to\infty} x(t) = 1$

Theorem:

Let $\bar{x} \in \mathbb{R}^m$ be an equilibrium for an autonomous ODE

$$\frac{dx}{dt} = f(x) \qquad x \in \mathbb{R}^m$$

and let $V: U \to \mathbb{R}$ be a differentiable function defined on some neighbourhood U of $\bar{x}$ such that

(1) $V(\bar{x}) = 0$ and $V(x) > 0$ if $x \neq \bar{x}$

(2) $\frac{d}{dt} V(x(t)) \leq 0$ in U where $x(t)$ is soln of ODE

Then $\bar{x}$ is Lyapunov stable.

Moreover, if

(3) $\frac{d}{dt} V(x(t)) < 0$ for all $x(t) \in U \setminus \{\bar{x}\}$

then $\bar{x}$ is ~~Lyapun~~ asymptotically stable.

The above defined fn V is called a Lyapunov function

WARNING: there are no general methods to derive Lyapunov function for Lyapunov/asympt stable equilibria.

HOPE: there are classes of problems where there is a natural choice of Lyapunov function (physically motivated)

e.g. mechanical systems (possibly with dissipation/friction) have natural candidate $V = E$ (energy)

if $\frac{dE}{dt} = 0$ then local maxima and minima of energy E are Lyapunov stable equilibria.

for mechanical system with dissipation we have $\frac{dE}{dt} \leq 0$

---

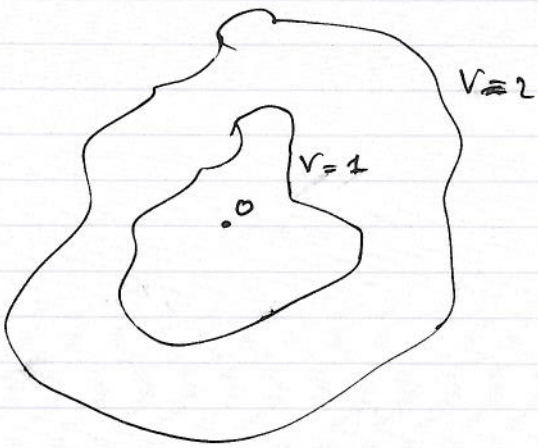
the picture (before we prove this theorem)

consider example in the plane $\mathbb{R}^2$ and consider $V : \mathbb{R}^2 \to \mathbb{R}$

$$V(0) = 0 \quad , \quad V(x) > 0 \quad \forall\, x \neq 0$$

natural to draw contourplot of V (curves at which V is constant)

$\frac{d}{dt} V(x(t)) \leq 0$



<u>two scenarios</u>: (1) along every soln V is strictly decreasing as fn of time $\Rightarrow \lim_{t\to\infty} x(t) = 0$

(2) the soln gets "stuck" on some level set of V so that $\lim_{t\to\infty} V(x(t)) = C > 0$

But in any case, we are Lyapunov stable because if $V(x(0)) = V_0$ then we can choose a small neighbourhood of $\bar{x} = 0$ such that in this neighbourhood $V < V_0$ (by continuity of V) $\Rightarrow$ any initial condition in this ball cannot intersect the level set $V_0 = V(x)$

If $\frac{d}{dt} V(x(t)) < 0$ then we cannot have $\lim_{t\to\infty} V(x(t)) = C > 0$

since this would imply that $\exists\, y$ (accumulated by $x(t)$) such that $\frac{d}{dt} V(x(t)) = 0$ if $x(0) = y$.

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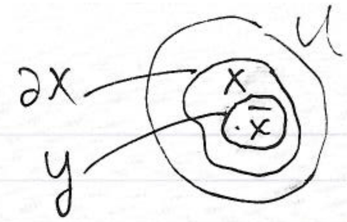
IV



Proof:

Let X be some nbh of $\bar{x}$ (equilibrium), with $X \subset U$.

Consider the function V on ∂X, the boundary of X

Since ∂X is closed and bounded and V is continuous in U

then the function V attains some minimum $V_{min}(\partial X) = \inf_{x \in \partial X} V(x)$

somewhere on ∂X

Now consider a smaller nbh Y of $\bar{x}$ such that

the maximum value of V on Y $\quad V_{max}(Y) = \inf_{x \in Y} V(x)$

is strictly smaller than $V_{min}(\partial X)$. The existence of Y is

guaranteed by continuity of V.

Hence, since $\frac{d}{dt} V(x(t)) \leq 0$ $\quad V$ cannot increase

along solution curves $\Rightarrow$ every sol. with initial condition at $t=0$

inside Y stays inside X for all positive time.

This construction holds for all nbhs X of $\bar{x}$ $\Rightarrow$

$\bar{x}$ is Lyapunov stable.

Suppose, in addition, that $\frac{d}{dt} V(x(t)) < 0$. We derive

a contradiction ~~suppose~~ to prove that $\lim_{t \to \infty} x(t) = \bar{x}$ $\forall x(0)$ in U with $V(x(0)) < V_{min}(\partial U)$.

Suppose $\lim_{t \to \infty} V(x(t)) = C > 0$ (i.e. $\lim_{t \to \infty} x(t) \neq \bar{x}$),

then there is a point y with $V(y) = C$ that is

accumulated by $x(t)$, ie. $\exists$ increasing sequence t_n $(n \to \infty)$

such that $\lim_{n \to \infty} x(t_n) = y$ $\Rightarrow$ $\frac{d}{dt} V(x(t)) = 0$ if $x(0) = y$. $\lightning$

exercise!