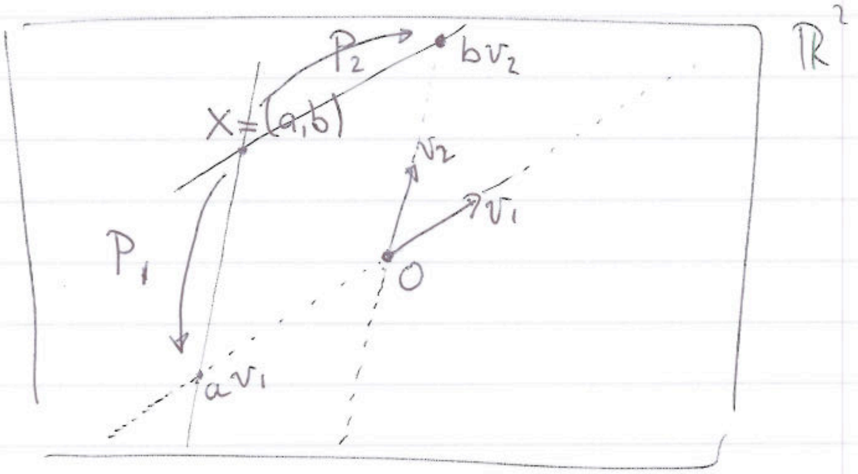
$\mathbb{R}^2 \ni x = a v_1 + b v_2$

$\phi^t(x) = a\phi^t(v_1) + b\phi^t(v_2)$

v_1 evec for eval λ_1 $\qquad \lambda_1 \neq \lambda_2$

v_2 evec for eval λ_2

$= a e^{\lambda_1 t} v_1 + b e^{\lambda_2 t} v_2$



<u>example:</u>

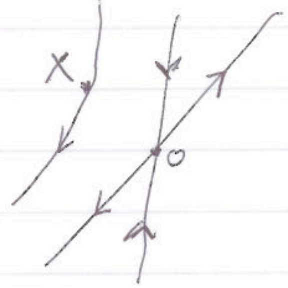
$\lambda_1 > 0$ $\qquad$ a coordinate is obtained by projecting x <u>to</u> $\langle v_1 \rangle$ <u>along</u> $\langle v_2 \rangle$

$\lambda_2 < 0$ $\qquad$ b proj x <u>to</u> $\langle v_2 \rangle$ <u>along</u> $\langle v_1 \rangle$

$\Downarrow$

expansion along $\langle v_1 \rangle$-axis
contraction along $\langle v_2 \rangle$-axis $\Big\}$ $t > 0$

So



Some solution curves

$$v = \begin{pmatrix} 1 \\ -i \end{pmatrix} \qquad \bar{v}^{\perp} = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$P_v = v \, \Big(\underbrace{\big(\overline{\bar{v}^{\perp}}\big)^T v} \Big)^{-1} \big(\overline{\bar{v}^{\perp}}\big)^T$$

$$\overline{\begin{pmatrix} i & 1 \end{pmatrix}} \begin{pmatrix} 1 \\ -i \end{pmatrix} = -2i \Rightarrow (-2i)^{-1} = \frac{i}{2}$$

$$v \cdot \big(\overline{\bar{v}^{\perp}}\big)^T = \begin{pmatrix} 1 \\ -i \end{pmatrix} \begin{pmatrix} -i & 1 \end{pmatrix} = \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}$$

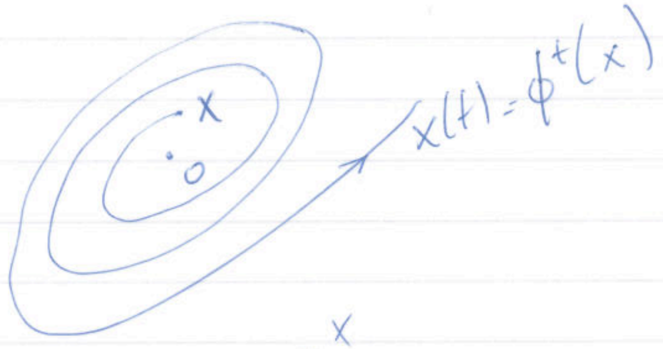
$$P_v = \begin{pmatrix} \frac{1}{2} & \frac{i}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{pmatrix}$$

---

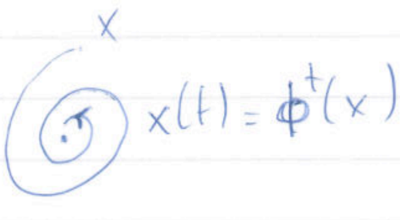
flow $\phi^t = e^{\alpha t} R(\beta t)$

↑ "rotation like"

$\alpha > 0$
$\beta \neq 0$



$\alpha < 0$
$\beta \neq 0$



$$A \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\exp\left(\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} t \right)$$

useful observation: $\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} = \alpha I + \beta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

these two matrices commute

$\Rightarrow$

now use that $\exp(A + B) = \exp(A) \cdot \exp(B)$

<u>iff</u> $AB = BA$

(i.e. if A and B commute)

if $AB \neq BA$ there is a well known formula dealing with the resulting expression
Baker-Campbell-Hausdorff formula.

$\Leftarrow$

so $\exp\left(\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} t \right) = \exp(\alpha I t) \exp\left(\beta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} t\right)$

$$= e^{\alpha t} \begin{pmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{pmatrix}$$

elementary calculation (check!).

$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ eigenvalues?

$\det(A - \lambda I) = 0$

$\det \begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix} = \lambda^2 = 0 \iff \lambda = 0$

algebraic multiplicity of $\lambda = 0$ eval is 2

geometric multiplicity = 1

$A \begin{pmatrix} x \\ y \end{pmatrix} = 0 \cdot \begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix} = 0$ iff $y = 0$.

eigenspace $\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$ with dimension 1

$\ker((A - \lambda I)^n)$ generalized eigenspace. ($n = 2$, $\lambda = 0$.)

$\ker(A^2)$

$A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\ker(A^2) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} = \mathbb{R}^2$

the whole $\mathbb{R}^2$ is generalized eigenspace of $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$